

# Chapter B

## One-Dimensional Kinematics

Blinn College - Physics 2325 - Terry Honan

Kinematics is the study of motion. This chapter will introduce the basic definitions of kinematics. The definitions of the velocity and acceleration will require the introduction of the basic notions of calculus, most specifically the derivative. We will also consider in detail the simple special cases of motion with constant velocity and constant acceleration. Free fall will be discussed as an example of motion with constant acceleration.

### B.1 - The General Problem

By one dimensional motion we mean motion constrained to a line. As examples, consider a car driving on a straight road or the vertical motion of an elevator. The problem of motion in two or three dimensions will be discussed in the next chapter.

#### Position as a Function of Time

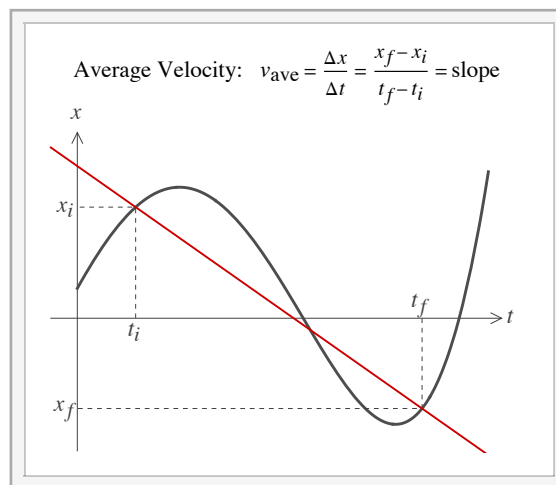
To mathematically define position we need to attach a real number line (the  $x$ -axis) to the line of motion. To do this there are two arbitrary choices; we must choose the  $x = 0$  position and then we must choose the positive direction.

Any 1D motion can be represented graphically. Time is the independent variable, so it will be the horizontal axis. We will then consider graphs of  $x$  as a function of time, where  $x$  is the vertical axis.

**Units:** The SI unit for distances is: m = meter

#### Velocity and the Derivative

##### Average Velocity



Interactive Figure

If a car drives 130 mi in 2 hours, we can calculate a velocity of 65 mi/hr. This is not necessarily what the speedometer would read; the speedometer reads the magnitude of the instantaneous velocity. In this case 65 mi/hr is what we call the average velocity.

We will define the average velocity by

$$v_{ave} = \frac{\Delta x}{\Delta t}$$

where  $\Delta$  (Delta) generally will represent the final value minus the initial value

$$\Delta x = x_f - x_i \text{ and } \Delta t = t_f - t_i.$$

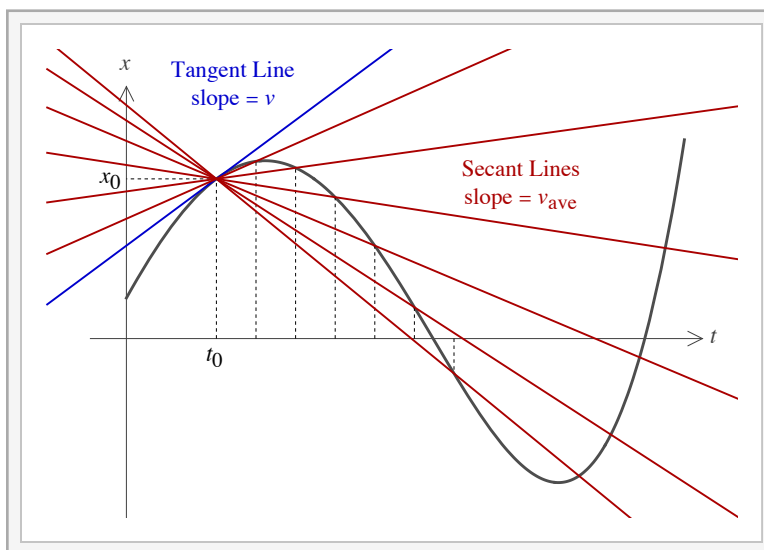
Note that the average velocity corresponds to two times  $t_i$  and  $t_f$ , and  $x_i$  and  $x_f$  are the positions at the two times. In a graph of  $x$  vs.  $t$  the average velocity has the interpretation as the slope of the secant line between the two points  $(t_i, x_i)$  and  $(t_f, x_f)$ .

**Units:** The SI unit for velocity is: m/s

The average velocity is not the distance per unit time, it is the displacement per unit time. The distinction is important. If you move along a line from an initial position to a different position a distance  $d$  away and then back to the initial position, your displacement is zero since  $\Delta x = x_f - x_i = 0$  but the total distance traveled is  $2d$ . We may define the average speed as the total distance divided by the total time.

$$\text{Average Speed} = \frac{\text{total distance}}{\text{total time}}$$

### Instantaneous Velocity



Interactive Figure

The instantaneous velocity refers to a single time  $t$ . Take the position at  $t$  to be  $x$ . We can then consider a later time  $t + \Delta t$ , where the position is  $x + \Delta x$ . The average velocity between these two times is  $\Delta x / \Delta t$ . To get the instantaneous velocity we let  $\Delta t$  become small; we do this by taking the limit as  $\Delta t \rightarrow 0$ . This gives the derivative of calculus; instantaneous velocity is the time derivative of position.

$$v = \frac{dx}{dt} = \dot{x} = \lim_{\Delta t \rightarrow 0} \frac{\Delta x}{\Delta t}$$

Most of our notation in calculus is due to its co-inventor Leibniz, but the dot notation introduced above is Newton's notation for time derivatives. Students should be aware of this notation because it is common in physics and engineering.

The graphical interpretation of the instantaneous velocity is simple. The average velocity is the slope of the secant lines. If we consider the secant lines corresponding to different  $\Delta t$  values then as  $\Delta t \rightarrow 0$ , these secant lines approach the tangent line. The velocity at  $t$  is then the slope of that tangent line. When we refer to the slope of a graph at some time, we mean the slope of the line tangent to the graph at that time.

The direction of motion in one dimension is given by the sign of the velocity. The speed is the velocity without its direction and is thus the absolute value of the velocity.

$$\text{speed} = |v|$$

### Acceleration

Acceleration is to velocity as the velocity is to the position. Velocity is the time derivative of position, so acceleration is the time derivative of the velocity.

#### Average Acceleration

Since the average velocity is related to the position by  $v_{\text{ave}} = \Delta x / \Delta t$  we can similarly write the average acceleration in terms of the velocity by

$$a_{\text{ave}} = \frac{\Delta v}{\Delta t}.$$

We can think of average acceleration graphically as the slope of the secant lines of a  $v$  vs.  $t$  graph.

**Units:** The SI unit for acceleration is:  $\text{m/s}^2$

### Instantaneous Acceleration

The instantaneous acceleration (or just acceleration) is the time derivative of the velocity.

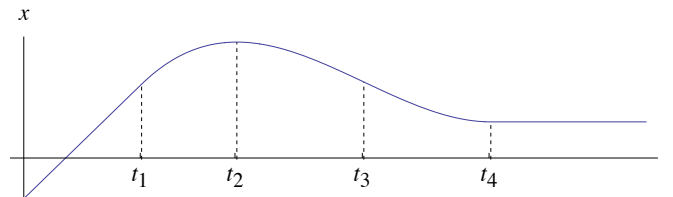
$$a = \frac{dv}{dt} = \dot{v} = \lim_{\Delta t \rightarrow 0} \frac{\Delta v}{\Delta t}$$

This can be written as the second derivative of the position.

$$a = \frac{d^2x}{dt^2} = \ddot{x}$$

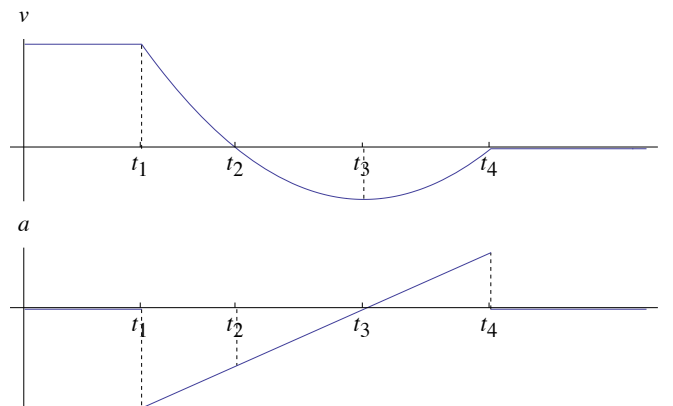
### Example B.1 - Graphical Analysis

Consider the following graph of  $x$  vs.  $t$ . For  $t < t_1$  the graph is a straight line. For  $t > t_4$  the graph is a horizontal line. Between  $t_1$  and  $t_4$  the function is cubic with a local maximum at  $t = t_2$  and an inflection point at  $t = t_3$ .



For this graph, plot  $v$  vs.  $t$  and  $a$  vs.  $t$ .

### Solution



**Discussion:** For  $t < t_1$  since the position-time graph is a line the velocity is a constant and constant  $v$  implies the acceleration is zero. For  $t > t_4$  the velocity has a constant zero-value and because  $v$  is constant the acceleration is also zero. Since the graph is cubic between  $t_1$  and  $t_4$  the velocity graph will be quadratic and the acceleration is a line. Because  $t_2$  is a local maximum of position, the velocity is zero. At the inflection point  $t_3$  the velocity is at its minimum and the acceleration becomes zero.

## Calculus Review

This section will be a brief review of the prerequisite calculus material.

### Differentiation

■ **Sum Rule:**  $\frac{d}{dt}(u + v) = \frac{du}{dt} + \frac{dv}{dt}$

- **Power Rule:**  $\frac{d}{dt} t^p = p t^{p-1}$  (Note  $\frac{d}{dt} t = 1$ )
- **Product Rule:**  $\frac{d}{dt} (u \cdot v) = \frac{du}{dt} \cdot v + u \cdot \frac{dv}{dt}$
- **Quotient Rule:**  $\frac{d}{dt} \left( \frac{u}{v} \right) = \frac{(du/dt) \cdot v - u \cdot (dv/dt)}{v^2}$
- **Chain Rule:**  $\frac{d}{dt} f[u(t)] = f'[u(t)] \cdot \frac{du}{dt}$  or  $\frac{dy}{dt} = \frac{dy}{du} \frac{du}{dt}$

### Antiderivatives and Indefinite Integrals

Antidifferentiation is the reverse of the differentiation procedure. If  $f(t)$  is a function then its antiderivative is another function  $F(t)$  that satisfies  $\frac{d}{dt} F(t) = f(t)$ . Generally, antidifferentiation is a more complicated procedure than differentiation. Since the derivative of a constant is zero, if  $F(t)$  is an antiderivative of  $f(t)$  then  $F(t) + C$  is also an antiderivative of  $f(t)$ , where  $C$  is an arbitrary constant. The result is stronger than this:  $F(t) + C$  is the *most general* antiderivative of  $f(t)$ . This general antiderivative is often written as an indefinite integral.

$$\int f(t) dt = F(t) + C \text{ where } F'(t) = f(t)$$

### The Definite Integral

Since  $v = dx/dt$  we can write the infinitesimal distance moved in the infinitesimal time  $dt$  as

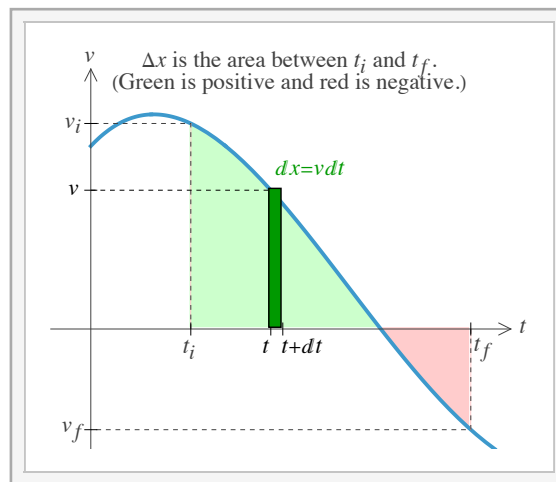
$$dx = v dt.$$

In a graph of  $v$  vs.  $t$  the area under the curve between  $t$  and  $t + dt$  is  $dx = v dt$ ; this is the area of a rectangle of height  $v$  and width  $dt$ . Thus the area under the graph over the infinitesimal time  $dt$  is the infinitesimal distance traveled in that time. The total distance traveled between the times  $t_i$  and  $t_f$  is  $\Delta x = x_f - x_i$ ; it is the sum of all the infinitesimal distances mentioned above, the total area under the curve. This is the definite integral of calculus.

$$\Delta x = \int_{t_i}^{t_f} v dt$$

Note the logic of this notation; in calculus  $d$  is implied to be a small  $\Delta$  and the sum over an infinite number of infinitesimal things becomes the integral.

$$\Delta \Rightarrow d \text{ and } \sum \Rightarrow \int$$



Interactive Figure

The definite integral generally has the interpretation as the area under a curve. The area under  $y = f(t)$  between  $a$  and  $b$  is written

$$\int_a^b f(t) dt$$

When the function is negative the contribution to the area is taken to be negative. Note how the velocity and position are related.  $v = dx/dt$ . The fundamental theorem of calculus is a generalization of this; it gives the rule we use to evaluate definite integrals.

$$\int_a^b f(t) dt = F(t) \Big|_a^b = F(b) - F(a) \text{ where } f(t) = \frac{d}{dt} F(t)$$

Thus, the definite integral of  $f$  is the difference of an antiderivative at the endpoints.

### Example B.2 - The Basic Definitions

The position as a function of time for a particle moving in one dimension is given in SI units by:

$$x(t) = 6t^3 - 8t^2 + 20.$$

(a) What is the average velocity between 1s and 3s?

#### Solution

The formula for average velocity is  $v_{\text{ave}} = \Delta x / \Delta t$ , so we need the position at each time.

$$x(1\text{s}) = 18\text{m} \text{ and } x(3\text{s}) = 110\text{m}$$

From this we can solve for the average velocity.

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{x(3\text{s}) - x(1\text{s})}{3\text{s} - 1\text{s}} = \frac{110\text{m} - 18\text{m}}{2\text{s}} = 46\text{m/s}$$

(b) What is the instantaneous velocity at 2s?

#### Solution

To find  $v(t)$  we must differentiate the position function.

$$v(t) = \frac{d}{dt} x(t) = 18t^2 - 16t$$

We then plug in the time.

$$v(2\text{s}) = 40\text{m/s}$$

(c) What is the average acceleration between 1s and 3s?

#### Solution

The formula for average acceleration is  $a_{\text{ave}} = \Delta v / \Delta t$ , so we need the velocity at each time.

$$v(1\text{s}) = 2\text{m/s} \text{ and } v(3\text{s}) = 114\text{m/s}$$

From this we can solve for the average acceleration.

$$a_{\text{ave}} = \frac{\Delta v}{\Delta t} = \frac{v(3\text{s}) - v(1\text{s})}{3\text{s} - 1\text{s}} = \frac{114\text{m/s} - 2\text{m/s}}{2\text{s}} = 56\text{m/s}^2$$

(d) What is the instantaneous acceleration at 2s?

#### Solution

To find  $a(t)$  we must differentiate the velocity function.

$$a(t) = \frac{d}{dt} v(t) = 36t - 16$$

We then plug in the time.

$$a(2\text{s}) = 56\text{m/s}^2$$

Note that the average acceleration for the interval equals the acceleration at the midpoint of the interval. This generally will not be the case but because our position function is cubic it turns out that this must be the case here.

(e) Find the net displacement  $\Delta x$  between 1s and 3s by integrating the velocity and compare with a direct calculation of  $\Delta x$ .

#### Solution

$$\begin{aligned}\Delta x &= \int_{t_i}^{t_f} v(t) dt = \int_{1s}^{3s} (18t^2 - 16t) dt \\ &= (6t^3 - 8t^2) \Big|_{1s}^{3s} = 90\text{m} - (-2\text{m}) = 92\text{m}\end{aligned}$$

Explicitly evaluating  $\Delta x$  gives the same:

$$\Delta x = x(3s) - x(1s) = 110\text{m} - 18\text{m} = 92\text{m}$$

## B.2 - Constant Velocity and Acceleration

Now that we have considered the general problem of one dimensional kinematics we can now consider special cases, first constant velocity, then constant acceleration. An important case of constant acceleration is free fall.

### Constant Velocity

If velocity is a constant then the acceleration is zero, since the derivative of a constant is zero. Let us now find the position from the velocity. Position is the antiderivative of the velocity.

$$\frac{dx}{dt} = v = \text{constant} \implies x(t) = vt + C \text{ where } C \text{ is a constant.}$$

Define the initial position  $x_0$  to be the position at  $t = 0$ ,  $x_0 = x(0)$ . Plugging this into our expression for  $x(t)$  gives  $C = x_0$  and

$$x(t) = x_0 + vt$$

If we choose the convention  $t_i = 0$ ,  $t_f = t$ ,  $x_i = x_0$  and  $x_f = x$  then we get  $\Delta x = x - x_0$ . The above expression becomes

$$x = x_0 + vt \text{ or } \Delta x = vt.$$

This is a simple expression; for constant velocity, the distance is the product of the velocity and time.

### Constant Acceleration

If the acceleration is a constant then to get the velocity we repeat the procedure for going from a constant velocity to the position. Velocity is the antiderivative of the acceleration.

$$\frac{dv}{dt} = a = \text{constant} \implies v(t) = at + C_1 \text{ where } C_1 \text{ is a constant.}$$

Define the initial velocity  $v_0$  to be the velocity at  $t = 0$ ,  $v_0 = v(0)$ . Plugging this into our expression for  $v(t)$  gives  $C_1 = v_0$  and

$$v(t) = v_0 + at$$

We need to antidifferentiate again to get the position as a function of time.

$$\frac{dx}{dt} = v(t) = v_0 + at \implies x(t) = v_0 t + \frac{1}{2} at^2 + C_2 \text{ where } C_2 \text{ is a different constant.}$$

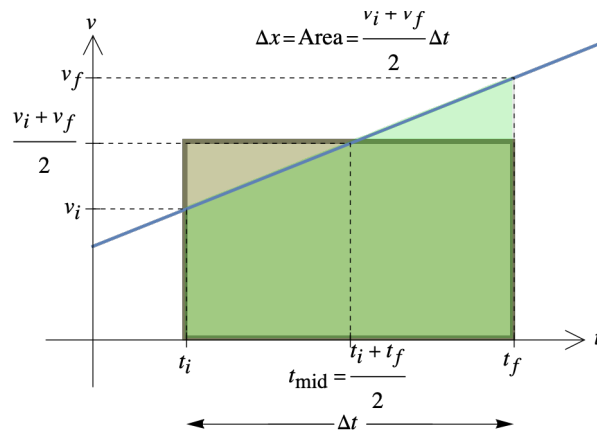
The arbitrary constant becomes the initial position  $x_0$  and we get

$$x(t) = x_0 + v_0 t + \frac{1}{2} at^2.$$

If we choose the same conventions as in the constant velocity case and add  $v_i = v_0$  and  $v_f = v$  then the above expressions for velocity and position become

$$v = v_0 + at \text{ and } \Delta x = v_0 t + \frac{1}{2} at^2.$$

Recall how to calculate  $\Delta x$  from  $v$  vs.  $t$ ; it is the area under the curve between  $t_i$  and  $t_f$ .



For constant acceleration the velocity vs. time is a straight line. We then get the area under a trapezoid with a base of width  $\Delta t$  and heights of  $v_i$  and  $v_f$ . This gives

$$\Delta x = \frac{1}{2} (v_i + v_f) \cdot \Delta t .$$

As a simple consequence of this we can see, for the case of constant acceleration, that the average velocity over some time interval is the average of the initial and final velocities.

$$v_{\text{ave}} = \frac{\Delta x}{\Delta t} = \frac{1}{2} (v_i + v_f)$$

Using our convention for  $v_0$ ,  $v$  and  $t$ , the above expression for  $\Delta x$  becomes

$$\Delta x = \frac{1}{2} (v_0 + v) t .$$

We now want to derive an expression relating  $\Delta x$ ,  $v_0$ ,  $v$  and  $a$ . To do this that the previous expression and  $v = v_0 + a t$ , and then eliminate time.

$$\Delta x = \frac{1}{2} (v_0 + v) \frac{v - v_0}{a} \implies v^2 - v_0^2 = 2 a \Delta x .$$

With this we have derived a set of four equations for kinematics with constant acceleration. These relate the variables  $t$ ,  $\Delta x$ ,  $v_0$ ,  $v$  and  $a$ . These will be useful for a large class of problems in this chapter.

### Constant Acceleration Equations

$$\begin{aligned} v &= v_0 + a t \\ \Delta x &= \frac{1}{2} (v_0 + v) t \\ \Delta x &= v_0 t + \frac{1}{2} a t^2 \\ v^2 &= v_0^2 + 2 a \Delta x \end{aligned}$$

#### Example B.3 - Decelerating Car

A car brakes uniformly to a stop from 30 m/s while moving 150 m.

(a) What is the car's acceleration while braking?

#### Solution

The first step is to establish what we are given and what we are looking for in terms of the variables in our constant acceleration equations. Here we are given the initial velocity, the final velocity, the displacement and we are looking for the acceleration.

$$v_0 = 30 \text{ m/s}, \quad v = 0, \quad \Delta x = 150 \text{ m} \quad \text{and} \quad a = ?$$

Often the best way to approach the constant acceleration equations is by identifying the equations in terms of the variable it does not include. Here we are not given time or looking for it so that leaves us to the fourth equation, the one that doesn't involve  $t$ .

$$v^2 = v_0^2 + 2 a \Delta x \implies a = -\frac{v_0^2}{2 \Delta x} = -3 \text{ m/s}^2$$

(b) How long does it take for the car to stop?

### Solution

Since we have already solved for  $a$  we can use any equation involving time to get  $t$ . Here will solve part (b) without reference to part (a); this leads us to the second equation, the one that does not involve  $a$ .

$$\Delta x = \frac{1}{2} (v_0 + v) t \implies t = \frac{2 \Delta x}{v_0} = 10 \text{ s}$$

## Free Fall

Free fall is one dimensional motion under the influence of only gravity. Assuming that only gravity acts implies that we are ignoring any friction effects. We will choose the convention that up is the positive direction. Also, we will take  $y$  as the position variable; this will be consistent with our later usage where  $y$  is typically taken as the upward vertical variable. Galileo discovered that the acceleration of all bodies in the presence of gravity (ignoring air resistance) is the same. The value of the downward acceleration is

$$g = 9.80 \frac{\text{m}}{\text{s}^2} = 32.0 \frac{\text{ft}}{\text{s}^2}$$

Since up is the positive  $y$  direction and the acceleration is downward we take the acceleration to be:

$$a = -g$$

Using this value of  $a$  and replacing  $x$  with  $y$  takes the constant acceleration equations to the free fall expressions.

### Free Fall Equations

$$\begin{aligned} v &= v_0 - g t \\ \Delta y &= \frac{1}{2} (v_0 + v) t \\ \Delta y &= v_0 t - \frac{1}{2} g t^2 \\ v^2 &= v_0^2 - 2 g \Delta y \end{aligned}$$

### Example B.4 - A Dropped Ball

A ball is dropped from a height  $h$ . (We will assume there is no air resistance for free-fall problems.)

(a) What is its time of fall?

### Solution

When solving a problem in terms of symbols, one must write the answer in terms of the symbols given and in terms of physical constants. Here what is given is  $h$  and the relevant constant is  $g$ .  $h$  is related to  $\Delta y$  and, since the net motion is downward,  $\Delta y < 0$ . Since the ball is dropped we conclude its initial velocity is zero. We are looking for  $t$ .

$$\Delta y = -h, \quad v_0 = 0 \quad \text{and} \quad t = ?$$

(Another way to understand  $\Delta y = -h$  is to set  $y_0 = h$ ,  $y = 0$  and use  $\Delta y = y - y_0$ .) The variable that we do not know or need is the final velocity  $v$ , so we are led to the third equation.

$$\Delta y = v_0 t - \frac{1}{2} g t^2 \implies -h = 0 - \frac{1}{2} g t^2$$

Note that generally for a dropped object, the time of fall is related to the height by

$$h = \frac{1}{2} g t^2.$$

Solving for  $t$  and taking the positive square root we get our answer.

$$t = \sqrt{\frac{2h}{g}}$$

(b) What is the ball's velocity when (just before) it hits the ground?

**Solution**

Now we are looking for  $v$ . To find this without reference to the result of part (a) we should use the fourth equation.

$$v^2 = v_0^2 - 2 g \Delta y$$

We still have  $\Delta y = -h$  and  $v_0 = 0$ , so we get:

$$v^2 = 2 g h .$$

The motion is downward so we want the negative square root when solving for  $v$ .

$$v = -\sqrt{2 g h}$$

If the problem asked for the speed instead of velocity then you would take the positive square root.

**Example B.5 - Upward Initial Velocity**

A ball is thrown straight upward from the ground at 30 m/s.

- (a) What is the maximum height reached by the ball?

**Solution**

We are given the initial velocity. How do we mathematically describe the ball's highest point? At the very top of its vertical path the ball is instantaneously at rest,  $v = 0$ . The maximum height  $y_{\max}$  is  $\Delta y$  when  $v = 0$ .

$$v_0 = 30 \text{ m/s}, v = 0, y_{\max} = \Delta y = ?$$

Since time is neither given or desired we are led to the fourth equation.

$$v^2 = v_0^2 - 2 g \Delta y \implies y_{\max} = \Delta y = \frac{v_0^2}{2 g} = 45.9 \text{ m}$$

- (b) How long does it take for the ball to return to the ground?

**Solution**

When the ball returns to where it began  $\Delta y = 0$ . Note that  $\Delta y$  is not the distance traveled; it is the net displacement. It is zero because  $\Delta y = y - y_0$  and  $y = y_0$ . Since we are solving for time and we still have  $v_0 = 30 \text{ m/s}$  we should use the third equation.

$$\Delta y = v_0 t - \frac{1}{2} g t^2 \implies 0 = t \left( v_0 - \frac{1}{2} g t \right) \implies t = 0 \text{ and } v_0 - \frac{1}{2} g t = 0$$

It is trivially true that  $\Delta y = 0$  when  $t = 0$ ; we want the other solution, which becomes:

$$t = \frac{2 v_0}{g} = 6.12 \text{ s}$$

- (c) Solve for  $t_{\text{up}}$ , the time for the ball to move to its highest point. Also show that when  $\Delta y = 0$ , as in the previous part,  $t = 2 t_{\text{up}}$ .

**Solution**

Since  $v = 0$  at the top, we have

$$v = v_0 - g t \implies 0 = v_0 - g t_{\text{up}} \implies t_{\text{up}} = \frac{v_0}{g} = 3.06 \text{ s}.$$

It follows that  $t = 2 t_{\text{up}}$ . This is generally the case for free-fall problems with  $\Delta y = 0$ .

- (d) When does the ball pass a 35 m high window?

**Solution**

We now want to find  $t$  when  $\Delta y = 35 \text{ m}$ . We still have  $v_0 = 30 \text{ m/s}$ , so the third equation is needed.

$$\Delta y = v_0 t - \frac{1}{2} g t^2 \implies 35 \text{ m} = (30 \text{ m/s}) t - \frac{1}{2} (9.80 \text{ m/s}^2) t^2$$

Solving for  $t$  gives two solutions. Both solutions are required here; it passes the window twice, moving upward and then moving downward.

$$t = 1.57 \text{ s and } t = 4.55 \text{ s}$$

(The above can be solved using the quadratic formula  $t = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$ , with  $a = \frac{1}{2} (9.80 \text{ m/s}^2)$ ,  $b = -30 \text{ m/s}$  and  $c = 35 \text{ m}$ .)

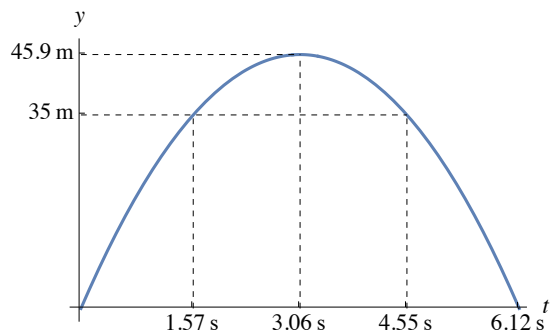


Figure -  $y$  vs.  $t$  graph showing the value calculated in this problem