Chapter G

Momentum and Systems of Particles

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G.1 - Momentum and Newton's Second Law

Definition of Momentum

So far we have written the second law in terms of the acceleration of a particle. It turns out that Newton wrote it differently; his preferred form of the second law was written in terms of the time derivative of the momentum of the body.

Momentum is a vector quantity describing the dynamics of a moving body. It is defined simply as the mass times the velocity and we use the symbol $\vec{p}$ to denote it.

$$\vec{p} = m\,\vec{v}$$

We will often refer to this as *linear momentum*; this will distinguish it from angular momentum which will be discussed later.

The Second Law

If we take the time derivative of the momentum then, using the product rule, we get

$$\frac{d}{dt}\vec{p} = \left(\frac{d}{dt}m\right)\vec{v} + m\left(\frac{d}{dt}\vec{v}\right) = m\,\vec{a} + \left(\frac{d}{dt}m\right)\vec{v}.$$

Here we are considering a more general possibility, that the mass of a body may change with time. In the case of a constant mass we get

$$\frac{d}{dt}m = 0 \implies \frac{d}{dt}\vec{p} = m\,\vec{a}.$$

This allows us to rewrite the second law in terms of the time derivative of the momentum.

$$\vec{F}_{\text{net}} = \frac{d}{dt}\vec{p}$$

This form is equivalent to the $m\,\vec{a}$ form when the mass is constant. When the mass is changing which form should we use? This new momentum form is the proper one.

G.2 - Impulse and Momentum

Impulse

Collisions are usually quick things but they are not instantaneous. When a bat hits a baseball, the ball rides along the bat for a period of time. An impulsive force is a large force acting over a short period of time. We will define the impulse as the integral of the force over time. If we take the collision to be between t_i and t_f the force is zero other than in that time interval.

Define the impulse as the integral of the force over the time of some collision.

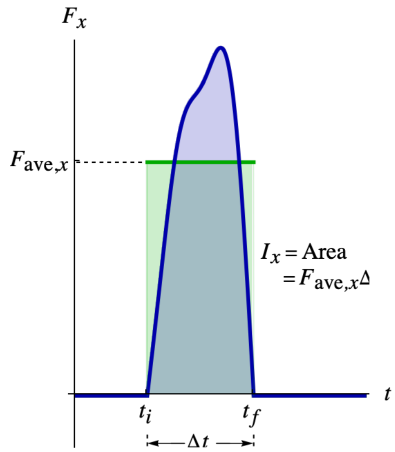
$$\vec{I} = \int_{t_i}^{t_f} \vec{F}\,dt$$

For a one dimensional force consider a graph of F vs. t. Since the definite integral is the area under a curve the impulse has the interpretation of the area under the force vs. time graph. For two or three dimensions the x-component of the impulse is the integral of the x-component of the force

Average Force

Generally, the average of a function over an interval is the integral of the function over the interval divided by the width of the interval. For the force the integral is the impulse and the width of the interval is $\Delta t = t_f - t_i$. The average force is

$$\vec{F}_{\text{ave}} = \frac{\int_{t_i}^{t_f} \vec{F}\, dt}{\Delta t} \quad \text{or} \quad \vec{I} = \vec{F}_{\text{ave}}\, \Delta t$$



The Impulse-Momentum Theorem

The impulse-momentum theorem is an immediate consequence of Newton's second law and the fundamental theorem of calculus.

$$\vec{I}_{\text{net}} = \Delta \vec{p}$$

Here the net impulse $\vec{I}_{\text{net}}$ is the impulse of the net force $\vec{F}_{\text{net}}$. Usually, the "net" subscript is omitted when writing the impulse-momentum theorem. The physical significance of this is when there is an impulsive force, it typically is much larger than any other forces acting over the short time of the collision and the other forces can be neglected. For example, when a bat hits a baseball, that force is much larger than gravity or some other force during the collision.

$$\vec{I}_{\text{net}} = \int_{t_i}^{t_f} \vec{F}_{\text{net}}\, dt = \int_{t_i}^{t_f} \frac{d}{dt} \vec{p}\; dt = \vec{p}(t_f) - \vec{p}(t_i) = \Delta \vec{p}$$

In the above proof, the first equality is the definition of impulse due to the net force. The second uses the second law and the third is the fundamental theorem of calculus.

$$\Delta \vec{p} = m \left(\vec{v}_f - \vec{v}_i\right)$$

Example G.1 - Hitting a Baseball

A 0.145 kg baseball thrown at 40 m/s is hit straight back at the pitcher at 50 m/s. If the bat is in contact with the ball for 0.035 s, then what is the average force of the bat on the ball?

Solution

As a vector the impulse-momentum theorem in one dimension becomes:

$$F_{\text{ave}}\, \Delta t = I = \Delta p = m \left(v_f - v_i\right).$$

Remembering that a one-dimensional vector is a real number, where the sign gives the direction we can write the given information as:

$$m = 0.145\ \text{kg},\ \ v_i = -40\ \text{m/s},\ \ v_f = 50\ \text{m/s}\ \ \text{and}\ \ \Delta t = 0.035\ \text{s}\ .$$

Because the ball changes direction, one of the velocities must be negative. If we choose the direction of the force of the bat on the ball to be positive then that makes the initial velocity negative. Solve for the average force.

$$F_{\text{ave}} = \frac{m}{\Delta t} \left(v_f - v_i\right) = 373\ \text{N}$$

G.3 - A System of Particles

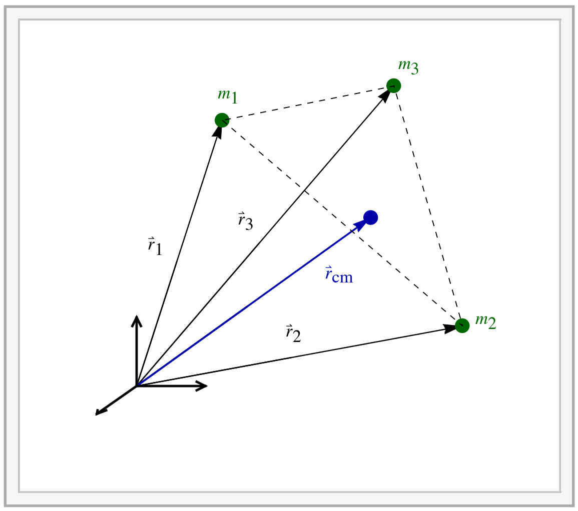
Introduction

So far our discussion of dynamics has applied only to point particles. When we discussed the dynamics of extended objects we treated them as particles. Why was this proper? In all the examples considered the body was not rotating, so each point on it had the same acceleration. This allowed us to treat it as a particle. If a body rotates then different points have different accelerations and we must be more careful. We can treat it as a system of particles.

A system is a collection of point particles. This could represent a huge number of particles, like every atom in a solid or fluid or it could be a small number like the earth, moon and sun. We arbitrarily divide the world into a system and everything else. We then break up the forces into internal forces, which are between particles of our system and external force between particles of our system and outside.

Center of Mass

Discrete Distribution



We will now define the center of mass of a system of particles as a weighted average of the positions. Consider the system to be a *discrete distribution*, that is a collection of point masses, which consist of masses m_1 at postion vector $\vec{r}_1$, m_2 at $\vec{r}_2$, etc. The total mass of our system is M:

$$M = m_1 + m_2 + \ldots = \sum_i m_i .$$

We define the position vector of the center of mass by

$$\vec{r}_{\text{cm}} = \frac{1}{M}\left(m_1\,\vec{r}_1 + m_2\,\vec{r}_2 + \ldots\right) = \frac{1}{M}\sum_i m_i\,\vec{r}_i .$$

Note that the position vectors $\vec{r}_i$ point from the origin of our coordinate system to the mass m_i. Since the x component of the position vector is just x, the x component of the center of mass is just

$$x_{\text{cm}} = \frac{1}{M}\left(m_1\,x_1 + m_2\,x_2 + \ldots\right) = \frac{1}{M}\sum_i m_i\,x_i ,$$

where the y and z components satisfy similar expressions.

Example G.2 - The Earth-Moon System

The masses of both the earth and moon and the earth-moon distance are given by

$$M_E = 5.97\times 10^{24}\ \text{kg},\quad M_M = 7.35\times 10^{22}\ \text{kg}\ \ \text{and}\ \ R_{EM} = 3.85\times 10^{8}\ \text{m} .$$

Where is the center of mass of the earth-moon system? Give its distance from the center of the earth.

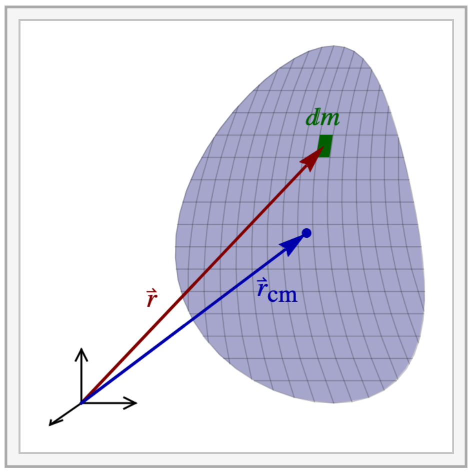
Solution

The center of mass must be along the line between the two centers. Take the origin to be at the center of the earth with the x-axis directed toward the moon.

$$x_{\text{cm}} = \frac{M_E\, x_E + M_M\, x_M}{M_E + M_M} = \frac{M_E\, 0 + M_M\, R_{EM}}{M_E + M_M} = \frac{M_M}{M_E + M_M}\, R_{EM}$$
$$= 0.0122\, R_{EM} = 4.68 \times 10^6 \text{ m}$$

Compare this to the earth's radius, $R_E = 6.38 \times 10^6$ m. The center of mass is below the earth's surface.

Continuous Distribution



Interactive Figure - Continuous Distribution

A *continuous distribution* is a solid object. To find its center of mass we can use calculus to break up the distribution into an infinite number of infinitesimal pieces. The infinitesimal mass of one of these infinitesimal pieces is dm. The vector from the origin to dm is $\vec{r}$. The total mass of the system M is found by summing over all the dm, where that sum becomes a definite integral.

$$M = \int dm$$

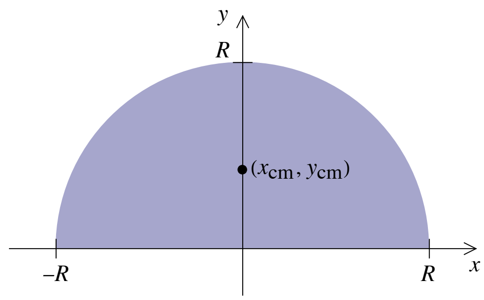
The sum giving the center of mass becomes:

$$\vec{r}_{\text{cm}} = \frac{1}{M} \int \vec{r}\, dm\,.$$

Example G.3 - Center of Mass of a Half Disk

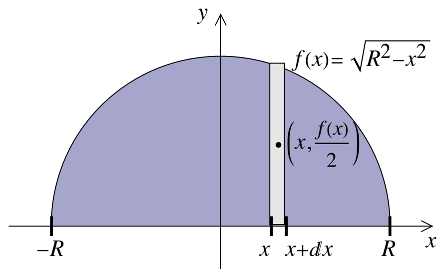
Find the coordinates of the center of mass of the uniform flat half disk described by

$$x^2 + y^2 \le R^2 \text{ and } y \ge 0\,.$$



Solution

Because of the symmetry in the diagram above it follows that $x_{cm} = 0$.



The half disk being uniform means that the mass is proportional to the area. For some infinitesimal area dA its infinitesimal mass is dm given by:

$$dm = \frac{M}{A_{tot}}\, dA = \frac{M}{\frac{1}{2}\pi R^2}\, dA$$

We need to break up the half disk into rectangles between x and $x + dx$ and vertically from $y = 0$ to $f(x) = \sqrt{R^2 - x^2}$. The infinitesimal area dA and infinitesimal mass dm become:

$$dA = f(x)\, dx \text{ and } dm = \frac{M}{\frac{1}{2}\pi R^2}\, dA = \frac{2M}{\pi R^2}\, f(x)\, dx\,.$$

We sum over all of these rectangles by integrating x with the limits

$$-R \le x \le R$$

The center of mass of each strip is $(x, f(x)/2)$. The coordinates of the center of mass are then given by integrating. Writing out the x-integral gives

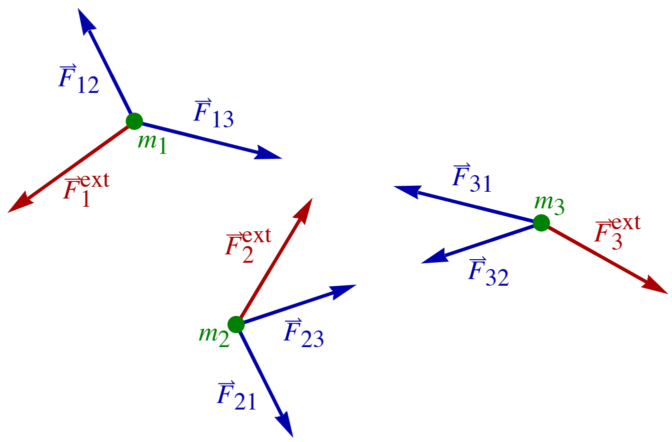
$$x_{cm} = \frac{1}{M}\int x\, dm = \frac{1}{M}\int_{-R}^{R} x\, \frac{2M}{\pi R^2}\, f(x)\, dx = \frac{2}{\pi R^2}\int_{-R}^{R} x\, \sqrt{R^2 - x^2}\, dx = 0.$$

The answer of 0 can be found by performing the integral but we have established earlier, by our symmetry argument, that it is 0. The integral for y_{cm} becomes

$$y_{cm} = \frac{1}{M}\int \frac{f(x)}{2}\, dm = \frac{1}{M}\int_{-R}^{R} \frac{f(x)}{2}\left(\frac{2M}{\pi R^2}\right) f(x)\, dx = \frac{1}{\pi R^2}\int_{-R}^{R}\left(R^2 - x^2\right) dx$$

$$y_{cm} = \frac{1}{\pi R^2}\left(R^2 x - \frac{1}{3}x^3\right)\Bigg|_{-R}^{R} = \frac{1}{\pi R^2}\left(\frac{2}{3}R^3 - -\frac{2}{3}R^3\right) = \frac{4}{3\pi}R$$

A Three Particle System



Consider a three particle system with masses m_1, m_2 and m_3. For the forces on $m_1$can be written as a sum of internal forces $\vec{F}_{12}$ and $\vec{F}_{13}$ and external forces $\vec{F}_1^{\text{ext}}$, representing everything outside our system acting on m_1. The forces for m_2 and m_3 break up similarly giving

$$\vec{F}_{\text{net},1} = \vec{F}_1^{\text{ext}} + \vec{F}_{12} + \vec{F}_{13} = \frac{d}{dt}\vec{p}_1 = m_1\,\vec{a}_1$$
$$\vec{F}_{\text{net},2} = \vec{F}_2^{\text{ext}} + \vec{F}_{21} + \vec{F}_{23} = \frac{d}{dt}\vec{p}_2 = m_2\,\vec{a}_2$$
$$\vec{F}_{\text{net},3} = \vec{F}_3^{\text{ext}} + \vec{F}_{31} + \vec{F}_{32} = \frac{d}{dt}\vec{p}_3 = m_3\,\vec{a}_3.$$

To concentrate on the bulk motion of our system we sum over these expressions. The crucial point is that the internal forces cancel by Newton's third law. $\vec{F}_{12} + \vec{F}_{21} = \vec{0}$, $\vec{F}_{13} + \vec{F}_{31} = \vec{0}$ and $\vec{F}_{23} + \vec{F}_{32} = \vec{0}$.

$$\vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} + \vec{F}_3^{\text{ext}} = \frac{d}{dt}(\vec{p}_1 + \vec{p}_2 + \vec{p}_3) = m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + m_3\,\vec{a}_3$$

The General System

For a general system we define $\vec{F}_{\text{net}}^{\text{ext}}$ as the net force

$$\vec{F}_{\text{net}}^{\text{ext}} = \vec{F}_1^{\text{ext}} + \vec{F}_2^{\text{ext}} + \ldots = \sum_i \vec{F}_i^{\text{ext}},$$

$\vec{p}_{\text{tot}}$ as the total momentum and M as the total mass

$$\vec{p}_{\text{tot}} = \vec{p}_1 + \vec{p}_2 + \ldots = \sum_i \vec{p}_i$$
$$M = m_1 + m_2 + \ldots = \sum_i m_i.$$

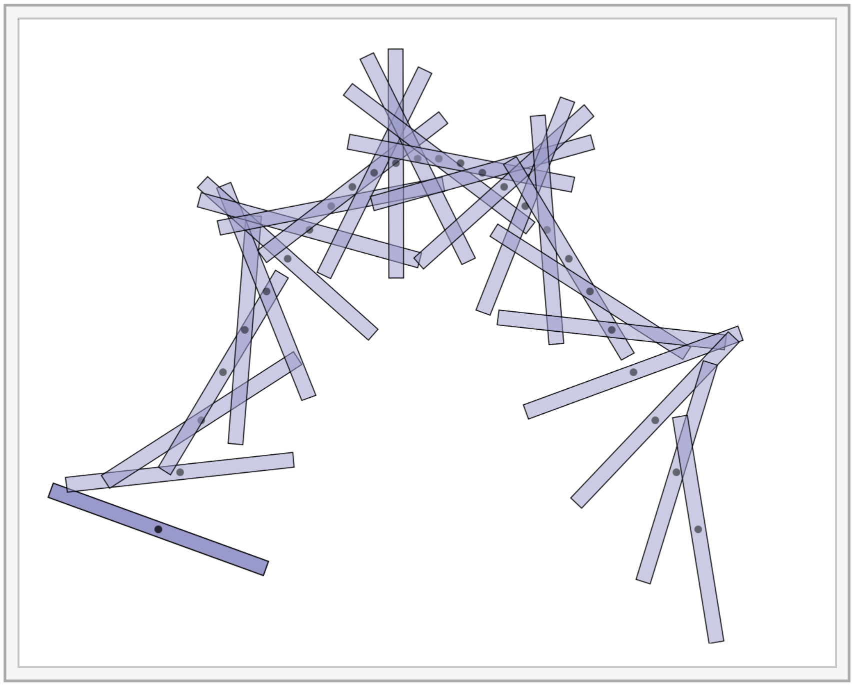
Taking two derivatives of our definition of the center of mass, $M\,\vec{r}_{\text{cm}} = m_1\,\vec{r}_1 + m_2\,\vec{r}_2 + \ldots = \sum_i m_i\,\vec{r}_i$, gives:

$$M\,\vec{a}_{\text{cm}} = m_1\,\vec{a}_1 + m_2\,\vec{a}_2 + \ldots = \sum_i m_i\,\vec{a}_i,$$

since the second derivative of the position vector $\vec{r}$ is the acceleration $\vec{a}$. With this we get the two expressions for the second law for a system of particles.

$$\vec{F}_{\text{net}}^{\text{ext}} = \frac{d}{dt}\vec{p}_{\text{tot}}$$
$$\vec{F}_{\text{net}}^{\text{ext}} = M\,\vec{a}_{\text{cm}}$$

Suppose a uniform stick is thrown so that it has both translational and rotational motion. The net external force acting on it is $\vec{F}_{\text{net}}^{\text{ext}} = M\,\vec{g}$, The second expression above implies that $\vec{a}_{\text{cm}} = \vec{g}$, so the stick's center of mass will follow the parabolic path of a projectile while the stick rotates about that path.



A stick thrown with both translational and rotational motion. The center of mass follows the arc of a projectile while the stick rotates about that.

Conservation of Linear Momentum

The conservation of momentum is a consequence of the momentum form of the second law for a system.

$$\vec{F}^{\text{ext}}_{\text{net}} = \frac{d}{dt}\vec{p}_{\text{tot}}$$

If there are no external forces acting on a system them the total momentum of the system is conserved.

$$\vec{F}^{\text{ext}}_{\text{net}} = \vec{0} \implies \frac{d}{dt}\vec{p}_{\text{tot}} = \vec{0} \implies \Delta\,\vec{p}_{\text{tot}} = \vec{0}$$

This is the second fundamentally conserved quantities encountered in our course. To see how this is fundamental, imagine enlarging the system to include everything; there is then, by definition, no external force and thus the total momentum is conserved.

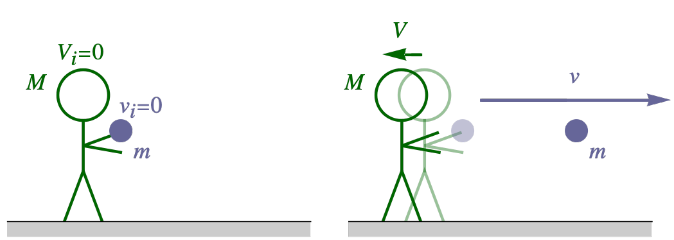
The conservation is also true component by component. If there is no external force in some direction, say the x direction, then the x component of the total momentum is conserved.

$$F^{\text{ext}}_{\text{net},x} = 0 \implies \frac{d}{dt}p_{\text{tot},x} = 0 \implies \Delta\,p_{\text{tot},x} = 0$$

Example G.4 - Throwing and Catching a Ball on Frictionless Ice

We will consider a man of mass M, initially at rest on perfectly frictionless ice.

(a) Suppose the man throws a ball of mass m forward at speed v. What is his recoil speed, V?

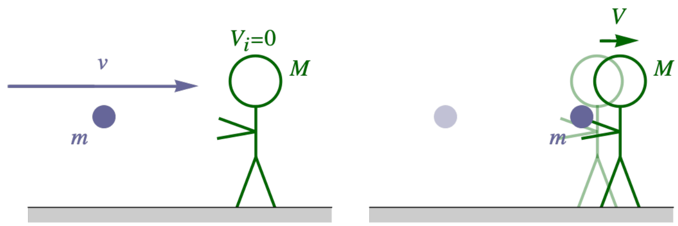


Solution

The man will recoil backward and his final velocity is negative, $-V$, where V is the speed. All velocities are horizontal here so we will take this to be a purely one-dimensional problem. We take the system to be the man and the ball, so there is no external force and the total momentum is conserved. We can then solve for the recoil speed.

$$F_{\text{net}}^{\text{ext}} = 0 \implies \Delta p_{\text{tot}} = 0 \implies p_{\text{tot},i} = p_{\text{tot},f} \implies 0 = M\,(-V) + m\,v \implies V = \frac{m}{M}\,v$$

(b) Suppose now that the man, still at rest initially, catches a ball of mass m thrown toward him. What is the final velocity of both the man and ball after the catch.



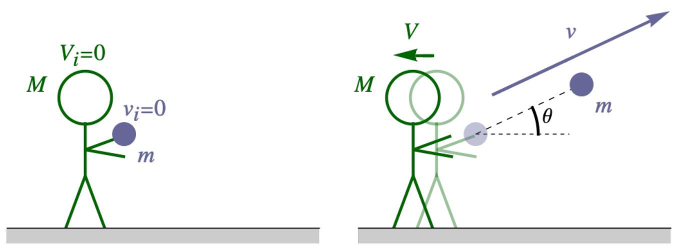
Initial Final

Solution

Again, we take this to be a purely one-dimensional problem and there is no external force on the man-ball system. The total momentum is still conserved. The final momentum will use the combined mass of $M+m$.

$$F_{\text{net}}^{\text{ext}} = 0 \implies \Delta p_{\text{tot}} = 0 \implies p_{\text{tot},i} = p_{\text{tot},f} \implies 0 + m\,v = (M+m)\,V \implies V = \frac{m}{M+m}\,v$$

(c) Now consider the two-dimensional modification of part (a) where the ball is thrown at an angle θ above vertical at speed v. Find the recoil speed, V.



Initial Final

Solution

To conserve momentum the man would need to recoil opposite the ball's velocity into the ice. Clearly this cannot happen because of the normal force of the ice on the man. The normal force is a net external force. Since there is no friction there is no horizontal net force, $F_{\text{net},x} = 0$, so the horizontal component of the total momentum is conserved. The horizontal component of the ball's velocity is $v_x = v\cos\theta$.

$$F_{\text{net},x}^{\text{ext}} = 0 \implies \Delta p_{\text{tot},x} = 0 \implies p_{\text{tot},x,i} = p_{\text{tot},x,f} \implies 0 = M\,(-V) + m\,v\cos\theta \implies V = \frac{m}{M}\,v\cos\theta$$

Symmetries and Conservation Laws - Noether's Theorem

It is a very deep and fundamental matter that symmetries give rise to conserved quantities. This result is known as Noether's theorem. The mathematician Amalie Noether demonstrated around 1920 that to every symmetry there is a conservation law. For example, the invariance or symmetry of the laws of physics under time translations, that the laws are the same now as a few minutes from now, implies that there is a conserved quantity; this is energy! The symmetry that the laws of physics are invariant under spatial translations implies a conserved quantity, in this case linear momentum. Rotational symmetry implies conservation of angular momentum.

The Center of Mass Frame and Energy

Taking the derivative of the definition of the center of mass gives

$$\vec{p}_{\text{tot}} = M\,\vec{v}_{\text{cm}}.$$

Recall the discussion of relative motion: $\vec{v}$ is the velocity of something with respect to a fixed frame, $\vec{v}'$ is the velocity with respect to a moving frame and $\vec{v}_0$ is the velocity of the moving frame. These are related by $\vec{v} = \vec{v}' + \vec{v}_0$. The center of mass frame is the frame that moves with the center of mass.

$$\vec{v}_0 = \vec{v}_{\text{cm}} \implies \vec{v} = \vec{v}' + \vec{v}_{\text{cm}}$$

In the center of mass frame it follows that $\vec{p}'_{\text{tot}} = \vec{0}$ and $\vec{v}'_{\text{cm}} = \vec{0}$. In other words, in the center of mass frame the center of mass is at rest.

The total kinetic energy of any system is

$$K_{\text{tot}} = \sum_i \frac{1}{2} m_i v_i^2$$

Let us now consider kinetic energy with respect to the center of mass frame. Here we use $\vec{v}_i = \vec{v}'_i + \vec{v}_{\text{cm}}$ to write $v_i^2 = v_i'^2 + v_{\text{cm}}^2 + 2\,\vec{v}_{\text{cm}} \cdot \vec{v}'_i$. Inserting this into the general expression for total kinetic energy gives

$$K_{\text{tot}} = \sum_i \frac{1}{2} m_i v_i'^2 + \sum_i \frac{1}{2} m_i v_{\text{cm}}^2 + \sum_i m_i\ \vec{v}_{\text{cm}} \cdot \vec{v}'_i\ .$$

We can factor the constant terms out of the sums giving

$$K_{\text{tot}} = \sum_i \frac{1}{2} m_i v_i'^2 + \frac{1}{2}\left(\sum_i m_i\right) v_{\text{cm}}^2 + \vec{v}_{\text{cm}} \cdot \sum_i m_i\, \vec{v}'_i\ .$$

Using $M = \sum_i m_i$ and $\vec{p}'_{\text{tot}} = \sum_i m_i \vec{v}'_i = \vec{0}$ we end up with a simple result.

$$K_{\text{tot}} = K'_{\text{tot,cm}} + \frac{1}{2} M v_{\text{cm}}^2$$

$$\text{where } K'_{\text{tot,cm}} = \sum_i \frac{1}{2} m_i v_i'^2 \text{ with } \vec{v}'_i = \vec{v}_i - \vec{v}_{\text{cm}}$$

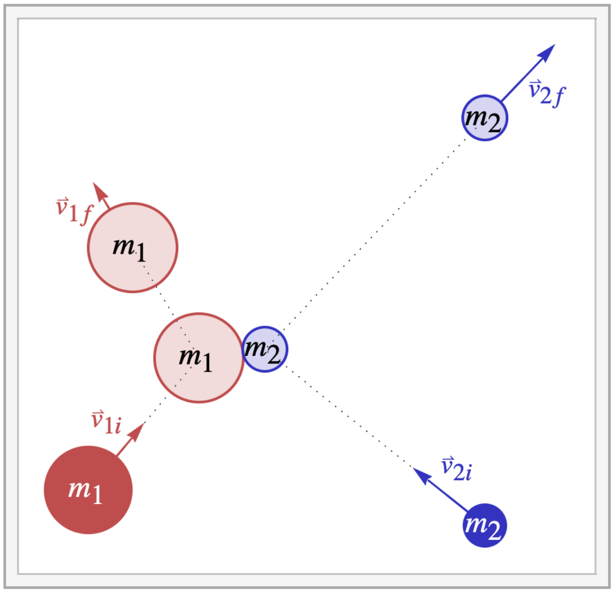
This expression is simple to interpret. $K'_{\text{tot,cm}}$ is the total kinetic energy in the center of mass frame. $(1/2)\,M\,v_{\text{cm}}^2$ is the energy of the bulk motion of the center of mass.

G.4 - Two-Body Collisions

Introduction

We now consider two-body collisions as a special case of our more general discussion. If there are no external forces act while two bodies collide, then the total momentum of the two-body system is conserved. Even if there are external forces, often we can neglect them and consider momentum conserved. Consider a mid-air collision between two bodies. Gravity acts as an external force during the collision but usually, to a good approximation, the collision is so fast that the large impulsive internal forces dominate the gravity force and gravity can be neglected. We can equate the total momentum just before and just after the collision.

Momentum Conservation



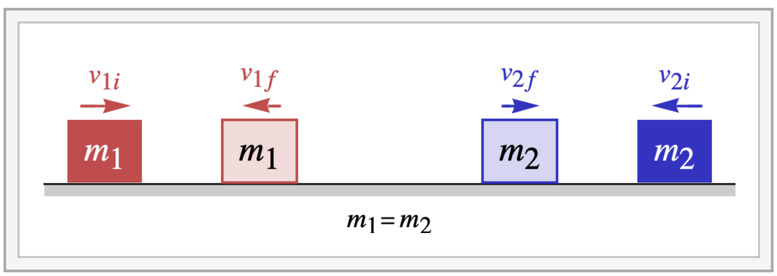
Interactive Figure

Mass m_1 moving at $\vec{v}_{1i}$ collides with mass m_2 moving at $\vec{v}_{2i}$. After the collision the velocities are $\vec{v}_{1f}$ and $\vec{v}_{2f}$. The conservation of momentum for a two-body collision has the form

$$m_1\,\vec{v}_{1i} + m_2\,\vec{v}_{2i} = m_1\,\vec{v}_{1f} + m_2\,\vec{v}_{2f}.$$

The left-hand side is the total initial momentum and the right hand side is the total final momentum.

In the case of a one dimensional collision then the above expression applies but we may omit the vector arrows. In one dimension a vector is a real number and the sign gives the direction. The vector nature of momentum and velocity is reflected in their signs.



Interactive Figure

Elastic Collisions - Kinetic Energy Conservation

Typically energy is lost in a collision. Often to a reasonable approximation we can consider conservation of energy. The relevant energy is kinetic.

$$\frac{1}{2}\,m_1\,v_{1i}^2 + \frac{1}{2}\,m_2\,v_{2i}^2 = \frac{1}{2}\,m_1\,v_{1f}^2 + \frac{1}{2}\,m_2\,v_{2f}^2 \quad \text{(elastic)}$$

In other words

$$K_{\text{tot},i} = K_{\text{tot},f}. \quad \text{(elastic)}$$

Inelastic and Totally Inelastic Collisions

When we say a collision is inelastic we mean merely that it is not elastic.

$$K_{\text{tot},i} \neq K_{\text{tot},f} \quad \text{(inelastic)}$$

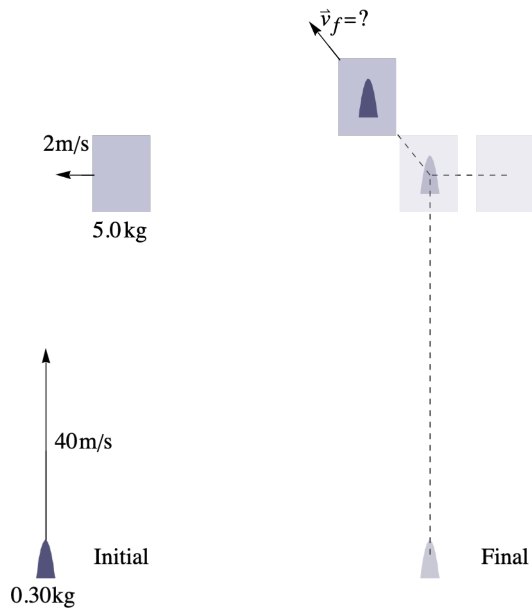
The extreme case of an inelastic collision is called totally inelastic. The most energy that can be lost in a collision is when the center of mass energy vanishes $K'_{\text{tot,cm}} = 0$. This means there is no relative motion in the center of mass frame. With two bodies this means that they have the same final velocity.

$$K'_{\mathrm{tot,cm}} = 0 \iff \vec{v}_{1f} = \vec{v}_{2f} = \vec{v}_f \text{ (totally inelastic)}$$

The conservation of momentum formula then has the simple form:

$$m_1\,\vec{v}_{1i} + m_2\,\vec{v}_{2i} = (m_1 + m_2)\,\vec{v}_f. \text{ (totally inelastic)}$$

Example G.5 - A Projectile embeds in a Block



(a) A projectile with a mass of 0.30 kg moving in the y-direction at 40 m/s collides with and embeds in a 5.0-kg mass moving in the negative x-direction at 2.0 m/s. What is the combined final velocity of the block and projectile after the collision?

Solution

$$m_1 = 0.30\text{ kg},\ m_2 = 5.0\text{ kg},\ \vec{v}_{1i} = \langle 0, 40\rangle\text{ m/s},\ \vec{v}_{2i} = \langle -2.0, 0\rangle\text{ m/s}$$

This is a totally inelastic collision. We can solve for the final velocity.

$$m_1\,\vec{v}_{1i} + m_2\,\vec{v}_{2i} = (m_1 + m_2)\,\vec{v}_f \implies \vec{v}_f = \frac{m_1\,\vec{v}_{1i} + m_2\,\vec{v}_{2i}}{m_1 + m_2} = \langle -1.89, 2.26\rangle\text{ m/s}$$

(b) A one-dimensional version of the previous problem follows: A projectile with a mass of 0.30 kg moving in the x-direction at 40 m/s collides with and embeds in a 5.0-kg mass moving in the negative x-direction at 2.0 m/s. What is the combined final velocity of the block and projectile after the collision?

Solution

One-dimensional vectors are real numbers where the sign gives the direction. We must be careful with signs.

$$m_1 = 0.30\text{ kg},\ m_2 = 5.0\text{ kg},\ v_{1i} = 40\text{ m/s},\ v_{2i} = -2.0\text{ m/s}$$

We can similarly solve for the final velocity.

$$m_1\,v_{1i} + m_2\,v_{2i} = (m_1 + m_2)\,v_f \implies v_f = \frac{m_1\,v_{1i} + m_2\,v_{2i}}{m_1 + m_2} = +0.377\text{ m/s}$$

One Dimensional Elastic Collisions

For the case of a one dimensional elastic collision we can solve for the final velocities in terms of the initial velocities and the masses.

$$m_1\,v_{1i} + m_2\,v_{2i} = m_1\,v_{1f} + m_2\,v_{2f} \quad \text{(momentum eq.)}$$

$$\frac{1}{2}\,m_1\,v_{1i}^2 + \frac{1}{2}\,m_2\,v_{2i}^2 = \frac{1}{2}\,m_1\,v_{1f}^2 + \frac{1}{2}\,m_2\,v_{2f}^2 \quad \text{(energy eq.)}$$

Consider a graph of v_{2f} vs. v_{1f}. With the masses and initial velocities given the momentum equation becomes a line and the kinetic energy equation becomes an ellipse. It follows that there are two solutions for (v_{1f}, v_{2f}). To find this one could use the momentum equation to solve for one variable in terms of the other and insert that into the kinetic energy equation. This gives a quadratic expression for the remaining unknown. With this we can, eventually, get the two solutions. Of the two solutions one is a trivial solution and will always be ignored. It is clear that

$$v_{1f} = v_{1i} \text{ and } v_{2f} = v_{2i}$$

is a trivial solution to our problem; it corresponds to the two masses never colliding and will be ignored. We are left with the one physically important solution. This solution method awkward and tedious. We will now derive a simplified method for solving such problems.

Take the kinetic energy equation, multiply by two and put the m_1 terms on the left and the m_2 terms on the right.

$$m_1 \left(v_{1i}^2 - v_{1f}^2\right) = m_2 \left(v_{2f}^2 - v_{2i}^2\right)$$

Similarly, put the momentum equation in the form with m_1 terms on the left and the m_2 terms on the right.

$$m_1 \left(v_{1i} - v_{1f}\right) = m_2 \left(v_{2f} - v_{2i}\right)$$

Now divide the second expression into the first. Using $a^2 - b^2 = (a - b)(a + b)$ we get

$$v_{1i} + v_{1f} = v_{2i} + v_{2f}. \quad \text{(1D elastic eq.)}$$

We then replace the quadratic kinetic energy expression with the above expression, which we will refer to as the one-dimensional elastic equation. To solve the problem we then use this equation with the momentum equation.

Example G.6 - One-Dimensional Elastic Collision

(a) A car rolls at speed v toward a truck with twice the car's mass rolling in the opposite direction at the same speed. The collision is elastic and head-on, so that both vehicles stay on the same line. What are both final velocities?

Solution

We must write our answers in terms of the speed v. Since the car and truck are moving in opposite directions their velocities must have opposite signs. We will take the car's direction as positive. Take m as the mass of the car, so the track has mass $2m$. Since m is not given the answer cannot depend on it; we will see is cancels.

$$m_1 = m, \; m_2 = 2m, \; v_{1i} = v, \; v_{2i} = -v$$

The momentum conservation expression gives this expression:

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \implies m v + 2m(-v) = m v_{1f} + 2m v_{2f} \implies -v = v_{1f} + 2 v_{2f}$$

The 1D elastic equation gives

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \implies v + v_{1f} = -v + v_{2f} \implies 2v + v_{1f} = v_{2f}$$

Inserting this expression for v_{2f} into the momentum equation gives us v_{1f}.

$$v_{2f} = 2v + v_{1f} \implies -v = v_{1f} + 2\left(2v + v_{1f}\right) \implies -5v = 3 v_{1f} \implies v_{1f} = -\frac{5}{3} v$$

We can then solve for v_{2f}.

$$v_{2f} = 2v + v_{1f} = 2v - \frac{5}{3} v = \frac{1}{3} v$$

Because the 1D elastic equation is linear we have a unique solution, since the momentum equation is also linear.

(b) To illustrate the utility of the trick of using the one-dimensional elastic equation we will now re-solve the problem the hard way, by explicitly using the energy expression.

Solution

As before we will use the momentum conservation expression which is the same.

$$m_1 v_{1i} + m_2 v_{2i} = m_1 v_{1f} + m_2 v_{2f} \implies -v = v_{1f} + 2 v_{2f}$$

Multiplying the energy conservation by two gives a quadratic expression.

$$\frac{1}{2} m_1 v_{1i}^2 + \frac{1}{2} m_2 v_{2i}^2 = \frac{1}{2} m_1 v_{1f}^2 + \frac{1}{2} m_2 v_{2f}^2 \overset{2\times}{\implies} m v^2 + 2m v^2 = m v_{1f}^2 + 2m v_{2f}^2 \implies 3 v^2 = v_{1f}^2 + 2 v_{2f}^2$$

Solve the momentum expression for v_{1f} and plug that into the energy expression to get an expression for v_{2f}.

$$v_{1f} = -v - 2\,v_{2f} \implies 3\,v^2 = \left(v + 2\,v_{2f}\right)^2 + 2\,v_{2f}^2 \implies 0 = 6\,v_{2f}^2 + (4v)\,v_{2f} + \left(-2\,v^2\right)$$

We use the quadratic formula to find v_{2f}.

$$v_{2f} = \frac{-b \pm \sqrt{b^2 - 4\,a\,c}}{2\,a} = \frac{-(4v) \pm \sqrt{(4v)^2 - 4 \times 6\left(-2\,v^2\right)}}{2 \times 6} = \frac{-4v \pm 8v}{2 \times 6}$$

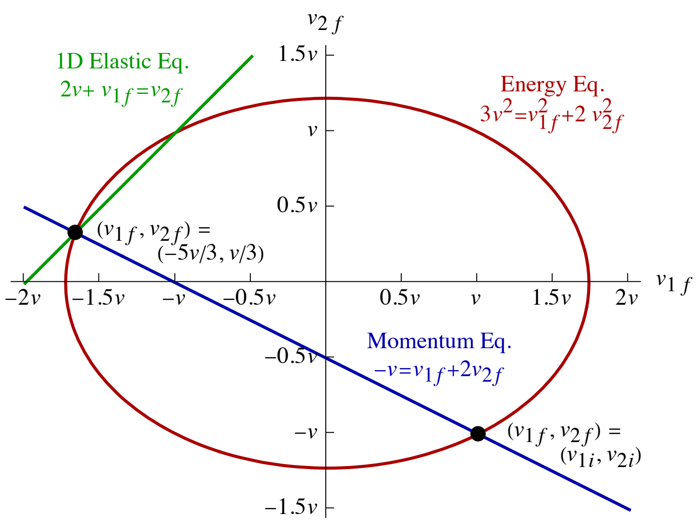
$$\implies v_{2f} = \frac{1}{3}\,v \text{ and } v_{2f} = -v$$

Using $v_{1f} = -v - 2\,v_{2f}$ we can find a v_{1f} value for each v_{2f} giving:

$$v_{1f} = -\frac{5}{3}\,v \text{ and } v_{1f} = v \text{ (respectively)}$$

The solution we are looking for uses the first value of v_{2f}, where the second is clearly the trivial solution.

The two solutions are displayed graphically in the v_{2f} versus v_{1f} plot below. Both the momentum equation and 1D elastic equation are linear and combine to give a unique solution. The energy equation plots as an ellipse; that ellipse and the line of the momentum equation have two solutions. The energy equation and the momentum equation always have a trivial solution where the final velocities equal the initial velocities: $v_{1f} = v_{1i}$ and $v_{2f} = v_{2i}$. This is simply the case where the two masses never collide. Omitting this trivial solution we get the same result as in part (a) but with considerably more work.



The 1D elastic equation and momentum equation have a unique solution but there are two solutions to the energy equation and the momentum equation. There is also the trivial solution where the final velocities equal their initial values.

Newton's Cradle

For a simple example of one-dimensional elastic collisions, consider Newton's cradle. First consider two identical steel balls of the same mass m where on is initially moving with speed v and the other at rest. The collisions between steel balls are elastic to a good approximation.

$$m_1 = m_2 = m\,, \quad v_{1i} = v \text{ and } v_{2i} = 0$$

We can now solve for the final velocities using the momentum conservation equation

$$m_1\,v_{1i} + m_2\,v_{2i} = m_1\,v_{1f} + m_2\,v_{2f} \implies m\,v + 0 = m\,v_{1f} + m\,v_{2f} \implies v = v_{1f} + v_{2f}$$

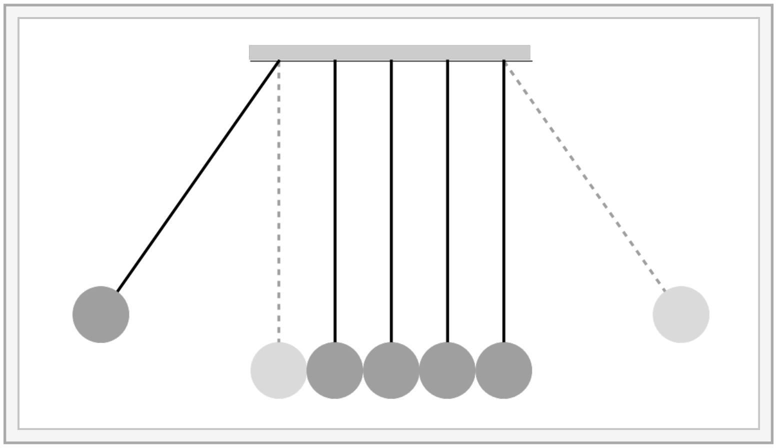
and the one-dimensional elastic equation

$$v_{1i} + v_{1f} = v_{2i} + v_{2f} \implies v + v_{1f} = 0 + v_{2f} \implies v + v_{1f} = v_{2f}$$

We can quickly solve for the solution.

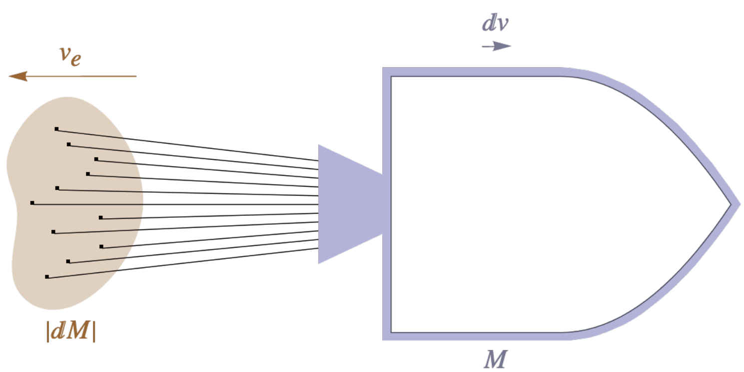
$$v_{1f} = 0 \text{ and } v_{2f} = v$$

So, m_1 stops and m_2 moves off with the same velocity that m_1 had before the collision. This explains the behavior of a simplified version of Newton's cradle with just two balls. The speed of the first ball just before the collision is determined by its initial release angle. With more balls there is a sequence of elastic collisions. The first ball with a speed v hits the second, the first stops and the second gains the speed v. The second ball instantly collides with the third ball which then gains the speed v. This continues until the last ball flies off with speed v and then bounces up to the same angle as the initial angle of the first ball.



G.5 - Rocket Propulsion

When moving with respect to a medium one can propel something forward by pushing backward. To walk, you push backward in the floor and the floor then pushes forward on you. A boat pushes backward on the water and the water pushes forward in it. A plane or jet propels itself similarly by pushing backward on the air. This leads to an obvious question for rocket propulsion. How does a rocket propel itself forward in the vacuum of space? The answer is that the rocket throws part of itself, its spent fuel, backward and thus propels the rest of the rocket forward. The mass change of a rocket is an essential part of its propulsion.



Consider a rocket of mass M moving with a velocity v. The rocket propels itself forward by shooting spent fuel backward at a speed of v_e, , the exhaust speed, relative to the rocket. In doing this the mass of the rocket changes. If dM is the infinitesimal change in the rocket's mass, then since $dM < 0$ the (positive) mass of ejected fuel is $|dM|$. Ejecting the fuel backward makes the rocket recoil forward by an infinitesimal dv. Looking at conservation of momentum in the frame where the rocket was initially at rest gives

$$0 = M\,dv - v_e\,|dM| = M\,dv + v_e\,dM \implies dv = -v_e\,\frac{dM}{M}.$$

We can integrate this expression and get

$$\int_{v_i}^{v_f} dv = -v_e \int_{M_i}^{M_f} \frac{dM}{M} \implies v_f - v_i = -v_e\left(\ln M_f - \ln M_i\right).$$

This leads us to our result relating the mass change of a rocket and the exhaust speed to the gain in velocity of the rocket.

$$v_f - v_i = v_e \ln \frac{M_i}{M_f}$$