

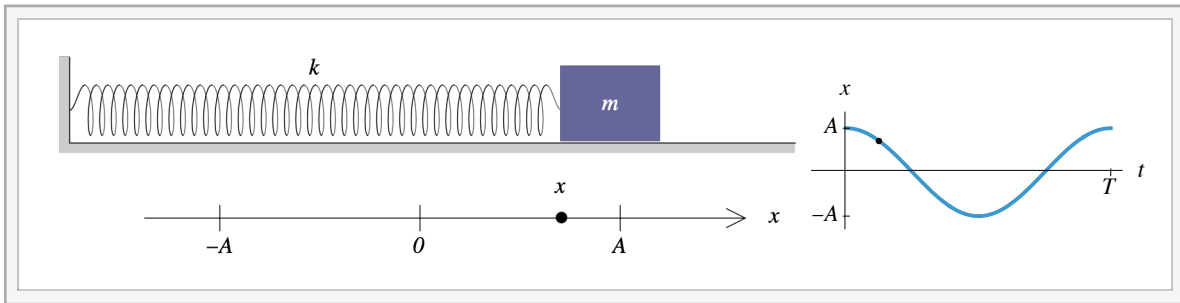
# Chapter K

## Oscillatory Motion

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### K.1 - Simple Harmonic Motion

#### The Mass-Spring System



Interactive Figure

Consider a mass sliding without friction on a horizontal surface. The force of a spring is given by Hooke's law  $F = -kx$ . Applying Newton's second law gives a second order ordinary differential equation (ODE)

$$F_{\text{net}} = ma \implies -kx = m \frac{d^2x}{dt^2}$$

Defining the angular frequency by

$$\omega = \sqrt{\frac{k}{m}},$$

gives the second order ODE

$$0 = \frac{d^2x}{dt^2} + \omega^2 x.$$

#### The Differential Equation for Simple Harmonic Motion

We need to distinguish our standard example of simple harmonic motion, the mass-spring system, from the general phenomenon. Generally, simple harmonic motion occurs any time a mechanical system gives rise to a differential equation of the form

$$0 = \frac{d^2x}{dt^2} + \omega^2 x.$$

We will see that this is a quite generic expression that occurs in situations that give oscillatory motion.

#### Comments on ODEs (Ordinary Differential Equations)

A differential equation (DE) is some equation involving a function and its derivatives.

The differential equation is solved to find the function.

The order of a DE is the highest number of derivatives.

If there is at most a second derivative it is a second order equation.

If it is a function of one variable it is an ordinary differential equation (ODE). For functions of several variables there are partial differential equations (PDE).

For functions of more than one variable we take partial derivatives instead of ordinary ones. The differential equations course (taken after Cal. III) is on ODEs.

The general solution of a  $p^{\text{th}}$  order ODE is any solution involving  $p$  independent arbitrary constants.

It is easy to verify that something is a solution to a differential equation; it is just a matter of taking derivatives and plugging into an equation. If a solution has the correct number of arbitrary constants then we can conclude that this is the general solution.

## The General Solution of the Differential Equation

We will now solve the ordinary differential equation (ODE) by guessing a solution.

$$0 = \frac{d^2 x}{dt^2} + \omega^2 x.$$

Since the second derivatives of both sine and cosine are the negatives of themselves, it follows that

$$\cos(\omega t) \text{ and } \sin(\omega t)$$

are each solutions to the differential equation. Because the ODE has the simple properties (a homogeneous linear equation) that:

- (i) a constant times a solution is a solution and
- (ii) the sum of two solutions is a solution.

it follows that

$$x(t) = B \cos(\omega t) + C \sin(\omega t)$$

is a solution, where  $B$  and  $C$  are arbitrary constants. Since we have solution to a second order ODE with two arbitrary constants, we can conclude that this is the general solution. We can interpret the constants  $B$  and  $C$  in terms of the initial position  $x_0$  and the initial velocity  $v_0$ .

$$\begin{aligned} x_0 = x(0) &= B \cdot 1 + C \cdot 0 = B \\ v(t) = \frac{d}{dt} x(t) &= -\omega B \sin(\omega t) + \omega C \cos(\omega t) \\ \implies v_0 = v(0) &= -\omega B \cdot 0 + \omega C \cdot 1 \implies C = \frac{v_0}{\omega} \end{aligned}$$

This gives

$$x(t) = x_0 \cos(\omega t) + \frac{v_0}{\omega} \sin(\omega t).$$

Another way of presenting this solution is

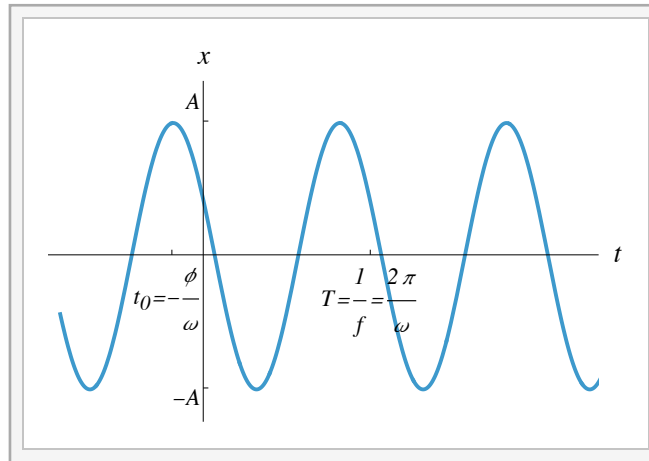
$$x(t) = A \cos(\omega t + \phi)$$

where the arbitrary constants are  $A$ , the amplitude, and  $\phi$ , the phase angle.  $\omega$  is called the angular frequency. This is related to the period  $T$  and the frequency  $f$ . The period is the time for one cycle. After one period the argument of the trig function shifts by  $2\pi$ .

$$t \rightarrow t + T \implies (\omega t + \phi) \rightarrow (\omega t + \phi) + 2\pi$$

This gives  $\omega T = 2\pi$ . The frequency is the number of cycles per time; since the time per cycle is the period we get  $f = 1/T$ . Combining this gives

$$T = \frac{1}{f} = \frac{2\pi}{\omega} \text{ and } f = \frac{1}{T} = \frac{\omega}{2\pi}.$$

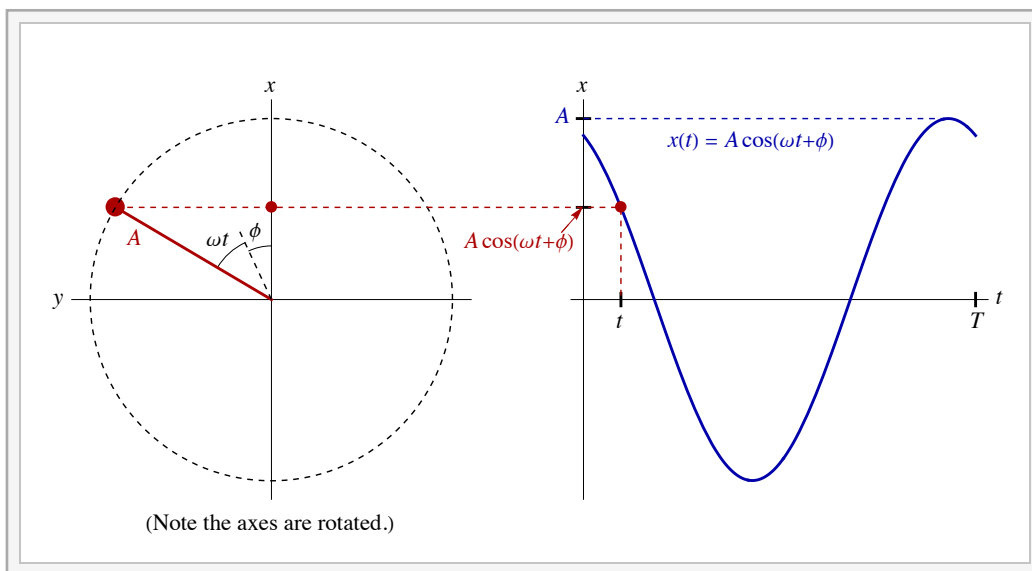


Interactive Figure

## Simple Harmonic Motion and Uniform Circular Motion

A way of understanding simple harmonic motion is by analogy to uniform circular motion. For polar coordinates we have:  $x = r \cos \theta$  and  $y = r \sin \theta$ . If moving in uniform circular motion then the radius  $r$  is a constant and the angle varies uniformly:  $\theta = \omega t + \theta_0$ . Take the initial angle  $\theta_0$  to be the phase angle  $\phi$  and the radius to be the amplitude  $A$ . We can then see that the  $x$ -component of uniform circular motion gives our solution for simple harmonic motion.

$$x(t) = A \cos(\omega t + \phi)$$



Interactive Figure - Simple harmonic motion can be viewed as one component of uniform circular motion.

## K.2 - Energy Considerations

### The Energy of a Mass-Spring System

The total energy is the sum of kinetic and potential energies. When the mass is at the turning points  $x = \pm A$ , its speed is zero; we can then write  $E = (1/2)kA^2$ . When it passes the equilibrium point  $x = 0$  the potential energy is zero and thus its kinetic energy is the maximum; so must the speed be its maximum,  $v = \pm v_{\max}$ . This allows us to write the energy with two equivalent forms for the total energy.

$$E = \frac{1}{2} m v^2 + \frac{1}{2} k x^2 = \begin{cases} \frac{1}{2} k A^2 \\ \frac{1}{2} m v_{\max}^2 \end{cases}$$

Equating the two expressions for the total energy gives an expression for the maximum speed in terms of the amplitude.

$$v_{\max} = \sqrt{\frac{k}{m}} A = \omega A$$

## Speed and Position for General Simple Harmonic Motion

For general simple harmonic motion we have

$$x(t) = A \cos(\omega t + \phi)$$

Taking the time derivative of this gives the velocity.

$$v(t) = -\omega A \sin(\omega t + \phi)$$

Since both sine and cosine vary between  $\pm 1$  we can identify the maximum speed as

$$v_{\max} = \omega A.$$

This is equivalent to what we had for the mass-spring case. The point here is to show that this is generally true. If the mass-spring energy is written in terms of the amplitude  $A$  then we can solve for  $v$  and, using the mass-spring value of  $\omega$ , get

$$v = \pm \omega \sqrt{A^2 - x^2}.$$

This expression is also generally true for simple harmonic motion. To verify that generally, we can write  $\cos(\omega t + \phi) = x/A$  and  $\sin(\omega t + \phi) = -v/(\omega A)$ . Using  $\cos^2 + \sin^2 = 1$  we can get the result.

### Example K.1 - Simple Harmonic Motion

A particle moves in simple harmonic motion with a frequency of 13 Hz. At  $t = 0$  the particle is released from rest from a distance of 2.4 cm from equilibrium.

(a) What is the maximum speed and maximum acceleration of the particle?

#### Solution

Since the particle is released from rest the amplitude is the initial distance from equilibrium.

$$A = 2.4 \text{ cm} = 0.024 \text{ m}$$

The angular frequency  $\omega$  is related to the frequency  $f$ .

$$f = 13 \text{ Hz} \implies \omega = 2\pi f = 81.681 \text{ s}^{-1}$$

The maximum speed can now be found.

$$v_{\max} = \omega A = 1.96 \text{ m/s}$$

Since the acceleration is the second time-derivative of the position the acceleration follows from the differential equation.

$$a = \frac{d^2 x}{dt^2} = -\omega^2 x$$

The magnitude of the maximum acceleration occurs at the largest value of  $x$ , which is the amplitude  $A$ .

$$a_{\max} = \omega^2 A = 160. \text{ m/s}^2$$

(b) At  $t = 0.22 \text{ s}$ , what are the position, velocity and acceleration of the particle?

#### Solution

Since at time zero the particle is at its maximum displacement we can conclude that the phase angle  $\phi$  is zero.

$$x(t) = A \cos(\omega t + \phi) = A \cos(\omega t)$$

Taking derivatives gives us the velocity and acceleration.

$$v(t) = \frac{dx}{dt} = -\omega A \sin(\omega t) \quad \text{and} \quad a(t) = \frac{dv}{dt} = -\omega^2 A \cos(\omega t)$$

At  $t_0 = 0.22$  s we get

$$x(t_0) = 0.0153 \text{ m}, \quad v(t_0) = 1.51 \text{ m/s} \quad \text{and} \quad a(t_0) = -102 \text{ m/s}^2$$

(c) When the particle is 1.7 cm from equilibrium, then what are the speed and the magnitude of the acceleration of the particle?

### Solution

The distance from equilibrium is  $|x|$ .

$$|x| = 1.7 \text{ cm} = 0.017 \text{ m}$$

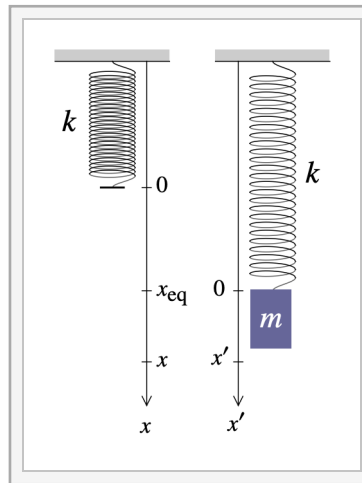
The speed is  $|v|$ .

$$|v| = \omega \sqrt{A^2 - x^2} = 1.38 \text{ m/s}$$

From the discussion in part (a) we get the magnitude of the acceleration.

$$|a| = \omega^2 |x| = 113 \text{ m/s}^2$$

## K.3 - The Vertical Mass-Spring



Interactive Figure

When a mass hangs from a spring we need to add the effect of gravity. As before, the second law gives our differential equation.

$$F_{\text{net}} = m a \implies -k x + m g = m \frac{d^2 x}{dt^2}$$

The equilibrium position of this is when the forces cancel  $F_{\text{net}} = 0$ . This gives

$$k x_{\text{eq}} = m g.$$

We can redefine our coordinates relative to the new equilibrium position.

$$x = x' + x_{\text{eq}}$$

If we insert this into our differential equation we get

$$-k x' = m \frac{d^2 x'}{dt^2}.$$

Here we have used the value of  $x_{\text{eq}}$  and the fact that the derivatives of constants vanish, which implies the second derivative of  $x$  is the same as the second derivative of  $x'$ .

The interpretation of the above expression is simple. The effect of gravity is trivial. It just shifts the equilibrium position and we end up with simple harmonic motion about the new equilibrium position.

### Example K.2 - Vertical Mass/Spring System

When a mass is hung from a vertical spring, the spring stretches by 8.5 cm. What is the period of oscillation of this system?

#### Solution

$$x_{\text{eq}} = 8.5 \text{ cm} = 0.085 \text{ m}$$

When hanging in equilibrium there are two forces acting on the mass, the spring force  $kx_{\text{eq}}$  acting upward and gravity downward. We do not know the mass  $m$  but we will see that it cancels; solve for  $k$  in terms of  $m$ .

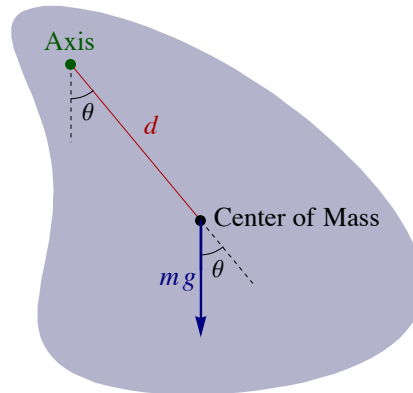
$$kx_{\text{eq}} = mg \implies k = \frac{mg}{x_{\text{eq}}}$$

From the angular frequency we can find the period.

$$\omega = \sqrt{\frac{k}{m}} = \sqrt{\frac{mg/x_{\text{eq}}}{m}} = \sqrt{\frac{g}{x_{\text{eq}}}} \implies T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{x_{\text{eq}}}{g}} = 0.585 \text{ s}$$

## K.4 - The Physical and the Simple Pendulum

### The Physical Pendulum



Consider a rigid body rotating without friction about an axis. The center of mass is a distance  $d$  from the axis. At equilibrium, the center of mass will hang below the center. Take the angle  $\theta$  to be the angle of the line from the axis to the center of mass measured from vertical; note that  $\theta = 0$  is the equilibrium position. The only nonzero torque acting on the rigid body is the torque due to gravity. This is

$$\tau_{\text{net}} = \tau_{\text{grav}} = -mgd \sin \theta.$$

The direction of positive  $\theta$  gives our sign convention for torque. The reason for the minus sign in the above expression is the torque tends toward smaller angles. Since the angular acceleration is the second time derivative of the angle, the rotational second law gives a differential equation.

$$\tau_{\text{net}} = I\alpha \implies -mgd \sin \theta = I \frac{d^2}{dt^2} \theta$$

$$\frac{d^2}{dt^2} \theta = -\frac{mgd}{I} \sin \theta$$

If we define

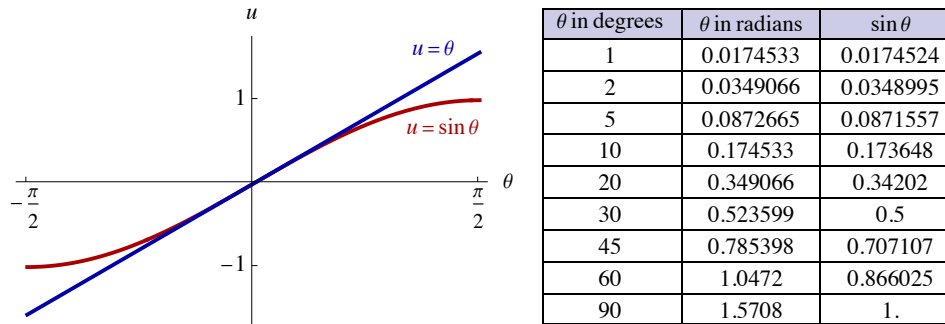
$$\omega = \sqrt{\frac{m g d}{I}}$$

then we get a differential equation of the form

$$\frac{d^2}{dt^2} \theta = -\omega^2 \sin \theta.$$

This is almost of the form of our simple harmonic motion equation  $d^2 x / dt^2 = -\omega^2 x$ , except for the sine function. If we consider small angles then we get

$$\sin \theta \approx \theta \text{ for small } \theta.$$



For small angles in radians, the sine of the angle is approximately equal to the angle in radians. This is shown with a graph on the left and with a table on the right.

We can then conclude that for small amplitude oscillations we have simple harmonic motion with an angular frequency  $\omega$  given by the expression above.

### Example K.3 - Physical Pendulum

A uniform meter stick swings without friction about a perpendicular axis through the 20 cm line. What is its period of small oscillations?

#### Solution

We use the parallel-axis theorem to find the moment of inertia.

$$I = I_{\text{cm}} + m d^2 = \frac{1}{12} m L^2 + m d^2$$

$m$  is not given but will cancel; the other values are given by

$$L = 1 \text{ m and } d = 30 \text{ cm} = 0.30 \text{ m}.$$

The period can then be found from the angular frequency.

$$\omega = \sqrt{\frac{m g d}{I}} \Rightarrow T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{I}{m g d}} = 2\pi \sqrt{\frac{L^2/12 + d^2}{g d}} = 1.53 \text{ s}$$

### The Simple Pendulum

The simple pendulum is a special case of the physical pendulum. It is the case where all the mass  $m$  is located at a point, the pendulum bob. If the bob is on the end of a string of length  $L$  then we get

$$d = L \text{ and } I = m L^2.$$

Solving for  $\omega$  we get

$$\omega = \sqrt{\frac{g}{L}}.$$

**Example K.4 - Texas A&M's Foucault Pendulum**

A Foucault Pendulum is a pendulum that precesses slowly in a circular path illustrating the earth's rotation underneath; this is due to the earth's Coriolis forces briefly mentioned in Chapter E. The George P. Mitchell Physics Building at Texas A&M University has a large Foucault Pendulum with a period of 10.32 s. What is the length of this pendulum?

**Solution**

$$\omega = \sqrt{\frac{g}{L}} \implies T = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{L}{g}} \implies L = g \left( \frac{T}{2\pi} \right)^2 = 26.4 \text{ m}$$