Chapter L

Elasticity and Waves

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Elastic Properties of Materials

Hooke's Law

Perfect rigidity is impossible. Under a force all materials will deform. To describe the elastic properties of materials will introduce stress and strain. Stress is related to the deforming force; for all types of deformations we will define stress as a force per area. We identify strain with the deformation, some dimensionless measure of it. When we introduced spring forces a force was said to be elastic when it satisfies Hooke's law; this means that the deforming force is proportional to the amount of deformation. In our current context Hooke's law becomes a proportionality between stress and strain.

$$\text{Stress} \propto \text{Strain} \qquad\qquad \text{(Hooke's Law)}$$

Most materials behave elastically for sufficiently small stresses and strains. With an elastic deformation materials return to their original shape. When stresses and strains exceed the elastic limit, then materials are permanently deformed; this is known as plasticity. At some point, sufficiently large stresses will lead to fracture.

To study the elastic properties of isotropic materials we will introduce three different elastic moduli, where each elastic modulus will be defined as

$$\text{Elastic Modulus} = \frac{\text{Elastic Stress}}{\text{Elastic Strain}}.$$

The different elastic moduli will be due to different types of stress and their associated deformations.

Stress and Pressure

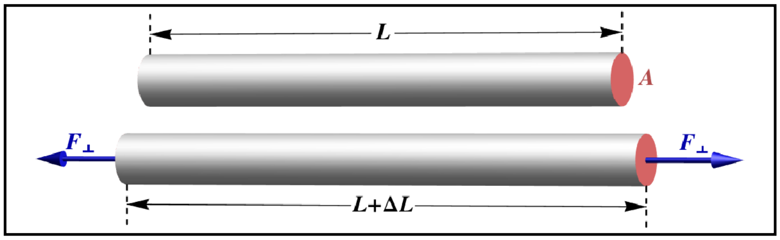
Imagine a situation of static equilibrium with some force $\vec{F}$ acting on a flat surface of area A of some material. $\vec{F}$ has components parallel and perpendicular to the surface. The perpendicular component $F_\perp$ stretches or compresses the material and the parallel components $F_\parallel$ alter the shape by skewing a rectangle into a parallelogram. Perpendicular forces are called tensile forces and $F_\perp / A$ is the tensile stress, where a negative tensile stress is a compression. Parallel forces are known as shear forces and $F_\parallel / A$ is the shear stress.

In a fluid there are no shear forces. The force is always perpendicular to the surface and, moreover, the force per area at some position is independent of the orientation of the surface. The pressure of water on the bottom of a pool corresponds to the weight of water above it, as is the case for a solid, but for water the pressure also pushes outward on the walls of the pool.

> **Units:** The SI unit for Pressure (Force/Area) is: pascal = Pa = N/m^2

Material	Young's Modulus $(\text{Pa} = \text{N}/\text{m}^2)$	Shear Modulus $(\text{Pa} = \text{N}/\text{m}^2)$	Bulk Modulus $(\text{Pa} = \text{N}/\text{m}^2)$
Steel	20×10^{10}	8.4×10^{10}	6×10^{10}
Aluminum	7.0×10^{10}	2.5×10^{10}	7.5×10^{10}
Copper	11×10^{10}	4.4×10^{10}	14×10^{10}
Lead	1.6×10^{10}	4.1×10^{10}	0.6×10^{10}

Young's Modulus



Young's modulus describes the change in length of a rod under a tensile force as shown above. The elastic stress is $F_\perp / A$ and the elastic strain is $\Delta L / L$. Young's modulus becomes

$$Y = \frac{F_\perp / A}{\Delta L / L}. \quad \text{(Young's Modulus)}$$

Note that under compression $F_\perp$ and ΔL become negative but the same value of Y applies.

We can find the effective spring constant of a rod or wire by solving for $F_\perp$. $F_\perp = (Y A / L) \Delta L$ gives $k_{\text{effective}} = Y A / L$.

Example L.1 - A Stretched Wire

When a 75 kg mass is hung on a copper wire, the wire stretches by 0.16%. What the the diameter of the wire?

Solution

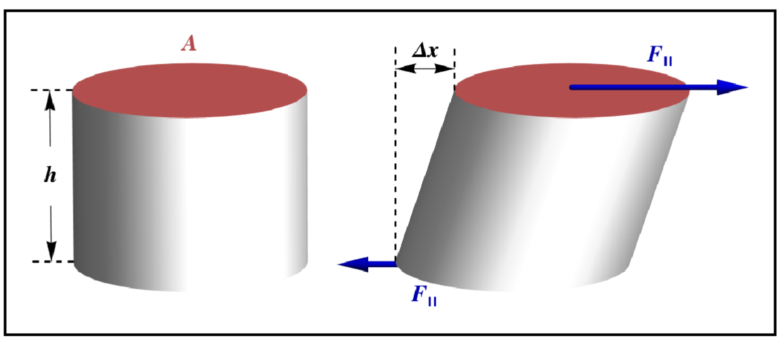
We know Young's modulus for copper and the fractional amount that the wire stretches. We also know the hanging mass and we need the value of g.

$$m = 75\text{ kg}, \quad g = 9.80\text{ m}/\text{s}^2, \quad Y = 11 \times 10^{10}\text{ Pa} \quad \text{and} \quad \Delta L / L = 0.0016$$

The definition of Young's modulus allows us to find the wire's cross-sectional area and then its diameter. The stretching force is the hanging weight $F_\perp = mg$.

$$Y = \frac{F_\perp / A}{\Delta L / L} \implies A = \frac{mg}{Y (\Delta L / L)} = 4.1761 \times 10^{-6}\text{ m}^2 = \pi \left(\frac{d}{2}\right)^2 \implies d = 1.15 \times 10^{-3}\text{ m}$$

Shear Modulus



The shear modulus describes the change in shape under a shear force as shown. The elastic stress is $F_{||} / A$ and the elastic strain is $\Delta x / h$. It is defined as

$$S = \frac{F_{||} / A}{\Delta x / h}. \quad \text{(Shear Modulus)}$$

Example L.2 - The Maximum Shear Stress of Steel

The maximum shear stress that steel can withstand is 4.0×10^8 Pa.

(a) What shear force with break a bolt with a diameter of 3.3 mm?

Solution

The diameter gives the area.

$$d = 3.3\times 10^{-3}\ \text{m} \implies A = \pi\left(\frac{d}{2}\right)^2 = 8.5530\times 10^{-6}\ \text{m}^2$$

Using σ for stress, we know can find the maximum shear force $F_{\parallel,\max}$ from the maximum shear stress and the area.

$$\frac{F_{\parallel,\max}}{A} = \sigma_{\max} = 4.0\times 10^{8}\ \text{Pa} \implies F_{\parallel,\max} = \sigma_{\max} A = 3420\ \text{N}$$

(b) What force is needed to punch a hole of diameter of 5.3 mm through a 0.80-mm thick steel sheet?

Solution

We are given the diameter of the hole d and the thickness of the sheet t. The force is a shear force that is parallel to the area of what becomes the rim of the hole. The relevant area is the circumference multiplied by the thickness.

$$d = 5.3\times 10^{-3}\ \text{m} \ \text{ and } \ t = 0.80\times 10^{-3}\ \text{m} \implies A = \pi d\times t = 4.24\times 10^{-6}\ \text{m}^2$$

Using σ for stress, we know can find the maximum shear force $F_{\parallel,\max}$ from the maximum shear stress and the area.

$$\frac{F_{\parallel}}{A} = \sigma_{\max} = 4.0\times 10^{8}\ \text{Pa} \implies F_{\parallel} = \sigma_{\max} A = 1700\ \text{N}$$

(c) What shear strain $\Delta x/h$ corresponds to the maximum shear stress $\sigma_{\max} = F_{\parallel,\max}/A$?

Solution

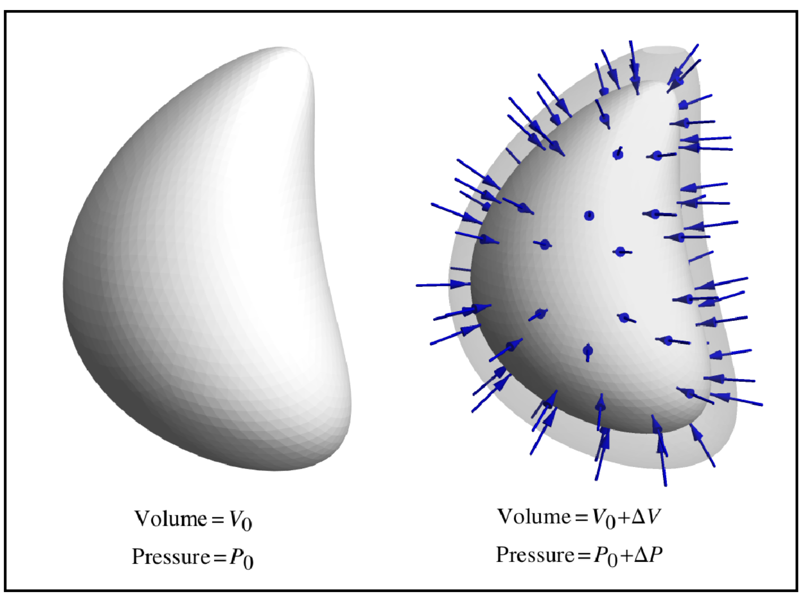
The shear modulus for steel is:

$$S = 8.4\times 10^{10}\ \text{Pa}$$

The shear strain follows from the definition of the shear modulus S.

$$S = \frac{F_{\parallel}/A}{\Delta x/h} = \frac{\sigma_{\max}}{\Delta x/h} \implies \frac{\Delta x}{h} = \frac{\sigma_{\max}}{S} = 0.00476 = 0.476\ \%$$

Bulk Modulus



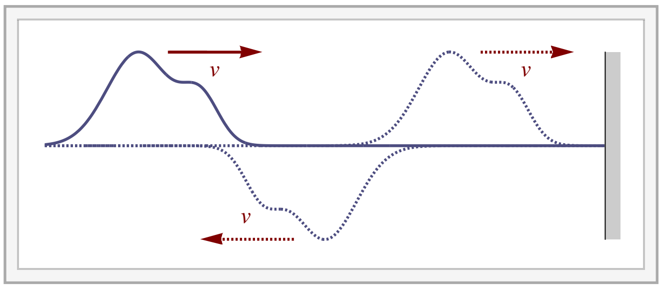
The bulk modulus describes the change in volume under a change in pressure ΔP as shown. The elastic stress is ΔP and the elastic strain is $\Delta V/V_0$. The bulk modulus then is defined as

$$B = -\frac{\Delta P}{\Delta V / V_0}. \quad \text{(Bulk Modulus)}$$

The minus sign is there to ensure that the modulus is positive, since an increase in pressure leads to a decrease in volume.

General Considerations for Waves

A mechanical wave is a disturbance in a medium that propagates through the medium. A point in the medium moves by a small amount from its equilibrium position but the wave transfers energy through the medium. We will generically label this disturbance by u. Consider a stretched string. Take x to be the position along the string and t is time. The equilibrium position of the string is its relaxed position. The disturbance y ($u = y$ for a string) is the perpendicular distance of a point on the string from equilibrium. When a pulse is put in the string it maintains its shape and travels the length of the string at a fixed speed. We will derive a formula for this speed; it will be $v = \sqrt{T / \mu}$, where T is the tension in the string and μ is the linear density of the string, its mass per length.



Interactive Figure

A wave is said to be *transverse* when the direction of the disturbance is perpendicular to the direction of propagation. Waves on a string are examples of transverse waves. Electromagnetic waves are also transverse; in the electromagnetic case the disturbance is the electric field, which is perpendicular to the direction of propagation. Note that electromagnetic waves are not mechanical waves; there is no medium and it can propagate in a vacuum. There is a plane of possible directions perpendicular to a direction of propagation. Choosing such a direction is choosing a *polarization*. Transverse waves can be polarized.

Consider a stretched spring. If a pulse is put into the spring then that pulse will propagate as a wave. Here a point on the spring moves back and forth a distance u from equilibrium but parallel to the direction of propagation x. When the disturbance is parallel to the direction of propagation we call the wave *longitudinal*. Longitudinal waves cannot be polarized.

Another example of a longitudinal wave is sound in a fluid. Here the molecules move back and forth parallel to the direction of wave propagation. We can view the disturbance of sound waves either in terms of displacement or in terms of pressure. Sound waves travel at the speed of sound. This varies with temperature; for air at 20 °C it is 343 m/s

Wave Type	**Disturbance – u**	**Transverse or Longitudinal**	**Wave Speed**
Waves on a string	$u = y(x, t)$	Transverse	$v = \sqrt{\frac{T}{\mu}}$
Electromagnetic Waves	$u = E =$ Electric field	Transverse	$v = c = 3.00 \times 10^8 \, \frac{\text{m}}{\text{s}}$
Compression Waves on a Spring	$u =$ Parallel disp.	Longitudinal	no formula given
Sound Waves in a Fluid	$u = P =$ Pressure or $u = s =$ Displacement	Longitudinal	$v = \sqrt{\frac{B}{\rho}}$ $v = 343 \, \frac{\text{m}}{\text{s}}$ (in air at 20 °C)
Longitudinal Sound Waves in a Solid	$u = s =$ Long. disp.	Longitudinal	$v = \sqrt{\frac{Y}{\rho}}$
Transverse Sound Waves in a Solid	$u = y =$ Trans. disp.	Transverse	$v = \sqrt{\frac{S}{\rho}}$

The One Dimensional Wave Equation

The Wave Equation

We will take some generic wave variable to be u which is some disturbance that varies as a function of x, the position, and t, time. For example, consider waves on a stretched string. The position along the string is labeled by x, t is time and u is the distance of a point on the string from its equilibrium position. The one dimensional wave equation is

$$\frac{\partial^2}{\partial x^2} u = \frac{1}{v^2} \frac{\partial^2}{\partial t^2} u,$$

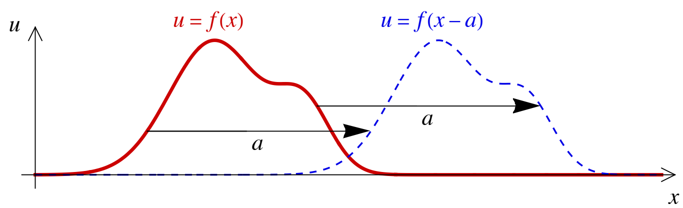
where v is the wave speed. This is a second order partial differential equation for $u(x, t)$.

The General Solution

The general solution of an ordinary differential equation (ODE), where we solve for functions of one variable, involves arbitrary constants. The general solution of a partial differential equation (PDE), where we are solving for functions of several variables, will involve arbitrary functions. The general solution is

$$u(x, t) = f(x - v\,t) + g(x + v\,t),$$

where f and g are arbitrary functions. To understand this general solution, consider the function $u(x) = f(x)$. If we shift this by a in the positive direction we get $u(x) = f(x - a)$. We can now see that $u(x, t) = f(x - v\,t)$ describes a pulse of arbitrary shape $u(x) = f(x)$ moving in the positive direction with speed v. $u(x, t) = g(x + v\,t)$ corresponds to a pulse of a different arbitrary shape moving in the opposite direction at the same speed.



To verify this is a solution we will plug $u(x, t) = f(x - v\,t)$ into the wave equation. We need to evaluate the partial derivatives using the chain rule:

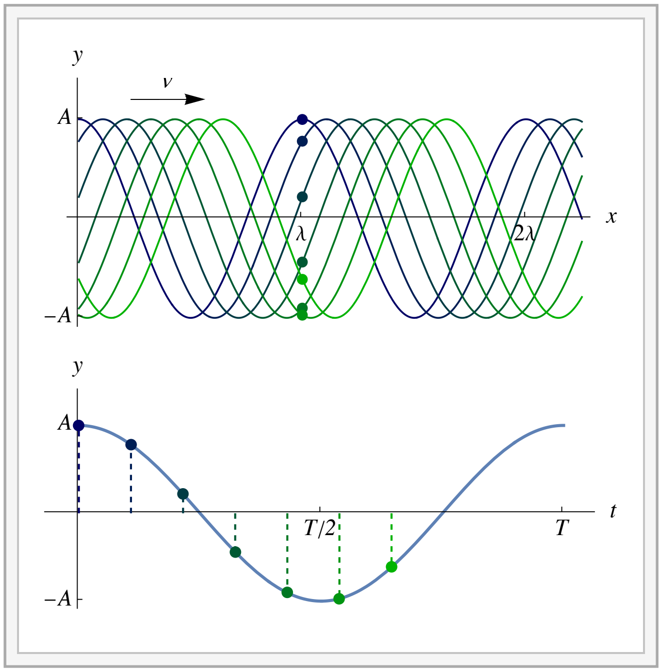
$$\frac{\partial^2}{\partial x^2} f(x - v\,t) = \frac{\partial}{\partial x} f'(x - v\,t) = f''(x - v\,t) \text{ and}$$

$$\frac{\partial^2}{\partial t^2} f(x - v\,t) = (-v) \frac{\partial}{\partial t} f'(x - v\,t) = (-v)^2 f''(x - v\,t).$$

Inserting $u = f(x - v\,t)$ into the wave equation we can see now that it is a solution for any function f. If we replace v with $-v$, it is still a solution and since f is arbitrary we can replace it with g; it follows that $u = g(x + v\,t)$ is also a solution. Since the derivative of the sum of two functions is the sum of the derivatives, the sum of our f and g solutions must also be a solution. This verifies that our expression for the general solution is indeed a solution. For it to be the general solution then *any* solution can be written in this form; to verify this is beyond the scope of the class.

If we take the graph $u = f(x)$ and shift it in the positive x-direction by a we get $u = f(x - a)$. This allows us to interpret our solution. The $u = f(x - v\,t)$ corresponds to a pulse of shape $u = f(x)$ moving in the positive direction with speed v and $u = g(x + v\,t)$ describes pulses of shape $u = g(x)$ moving in the negative x-direction with the same speed.

Sinusoidal Waves



Interactive Figure

We often consider waves where the shape of the pulse f (or g) are sinusoidal.

$$f(x) = A\cos(k\,x + \delta)$$

A is called the amplitude. k is called the wave number; this is related to the wavelength λ, which is the spatial period of the function. Since the period of sin is 2π and the period of f is λ we get $k\,\lambda = 2\pi$ or

$$k = \frac{2\pi}{\lambda}.$$

If we take this function f and move it in the positive or negative direction we get $f(x \mp v\,t) = A\cos[k\,(x \mp v\,t) + \delta]$ or

$$u(x, t) = A\cos(k\,x \mp \omega\,t + \delta),$$

where the angular frequency ω and wave number are related to the wave speed by $k\,v = \omega$. The argument of the cosine function $k\,x \mp \omega\,t + \delta$ is called the phase of a wave.

If we choose some point on the string x_0 then at that position moves as

$$A\cos(\omega\,t + \phi).$$

This is our expression for simple harmonic motion. Since the angular frequency is related to the frequency by $\omega = 2\pi f$ the wave speed can also be written in terms of the frequency and wavelength.

$$v = \frac{\omega}{k} = f\,\lambda$$

Dispersion and Generalized Waves

With our one-dimensional wave equation the speed of waves is independent of frequency. More generally, we can have different physical situations which are described by partial differential equations other than the one-dimensional wave equation (or its higher dimensional generalizations) but where we still get sinusoidal wave solutions. These generalized waves are *dispersive*, meaning that different frequency waves will have different speeds. Although we have sinusoidal solutions we no longer have simple pulses that maintain their shape as solutions. If we begin with a pulse then the pulse will spread out with time.

Light in a vacuum is a solution to a one or three-dimensional wave equation and are not dispersive. Light through a medium does have dispersion. A consequence of different frequencies, colors, having different speeds is the splitting of light into its spectrum by a prism. Another example of dispersive waves is surface water waves; these, it turns out, are a mixture of longitudinal and transverse displacements. In quantum mechanics matter waves, as described by the Schrödinger wave equation, are dispersive.

Waves on a String

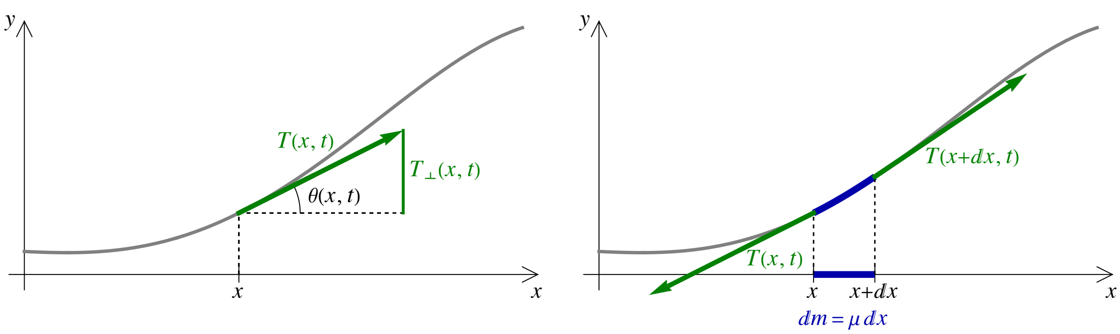
The Wave Equation and Speed

Consider a segment of small width δx between x and $x+\delta x$. Since μ is the mass per length, the small mass of this segment is $\mu\,\delta x$. Take the angle of the string from the x-axis to be θ. The tangent of the angle is the slope of the curve, $d y / d x$, but since y is a function of more than one variable we must use partial derivatives: $\tan\theta = \partial y / \partial x$. The assumption that our disturbance is small is equivalent to the assumption that y is small and thus the angle θ is small. For small angles we have:

$$\theta \simeq \sin\theta \simeq \tan\theta = \frac{\partial y}{\partial x}.$$

The tension in the string is T. We are interested in the transverse (perpendicular) motion of the string, where transverse means the y direction. The transverse component of the tension is

$$T_\perp = T\sin\theta = T\,\frac{\partial y}{\partial x}$$



We will apply the second law to a small segment of string of width δx. The tension pulls in opposite directions on either side of the small segment, so the net force for the segment will be the difference of the tensions at either end. Define $F_{\mathrm{net},\perp}$ to be the perpendicular component of the net force on the segment. We get:

$$F_{\mathrm{net},\perp} = T_\perp(x+\delta x,\, t) - T_\perp(x,\, t) = \frac{\partial T_\perp}{\partial x}\,\delta x = T\,\frac{\partial^2 y}{\partial x^2}\,\delta x\,,$$

where we have used that δx is small. Note that with our small angle approximation the component of the net force parallel to the string (the x component) vanishes.

The acceleration of the small segment is $\partial^2 y / \partial t^2$ and its small mass is $\mu\,\delta x$. Applying the second law gives:

$$F_{\mathrm{net},\perp} = m\,a_\perp \implies T\,\frac{\partial^2 y}{\partial x^2}\,\delta x = (\mu\,\delta x)\,\frac{\partial^2 y}{\partial t^2}.$$

We can rewrite this as

$$\frac{\partial^2 y}{\partial x^2} = \frac{1}{T/\mu}\,\frac{\partial^2 y}{\partial t^2}.$$

If we compare this with the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2}\,\frac{\partial^2 u}{\partial t^2},$$

we can see that with $u = y$ we get the wave equation with the speed given by

$$v = \sqrt{\frac{T}{\mu}}.$$

Power Carried by Waves on a String

Waves carry energy. When a sinusoidal wave travels down a string the flow of energy is steady. If some quantity of energy flows in a time then the energy per time or power is constant. Now consider some point on the string. If the wave is moving in some direction at the speed v, then the energy moves past the point at the same rate. In a short time δt all the energy in a small segment of width $\delta x = v\,\delta t$ will pass the point.

The energy of a particle in simple harmonic motion satisfies

$$E = \frac{1}{2}\,m\,v_{max}^2$$

Since a point on the string moves in simple harmonic motion we can apply this formula to δx. The mass of the small segment is

$$m = \mu\,\delta x = \mu\,v\,\delta t$$

and the energy in the small segment is, on average:

$$E_{ave} = \frac{1}{2}\,m\,v_{max}^2 = \frac{1}{2}\,\mu\,v\,\delta t\;v_{max}^2.$$

The power is given by $\mathcal{P}_{ave} = E_{ave}/\delta t$. The maximum speed of a point in simple harmonic motion is $v_{max} = \omega A$. It follow that the average power traveling down a string due to a sinusoidal wave is

$$\mathcal{P}_{ave} = \frac{1}{2}\,\mu\,A^2\,\omega^2\,v$$

Example L.3 - Waves on a Steel Wire

A wave of the form

$$y(x,t) = (0.020\text{ m})\sin\left[\left(105\text{ s}^{-1}\right)t + \left(3.0\text{ m}^{-1}\right)x\right]$$

travels down a steel wire with a linear density of 0.014226 kg / m.

(a) What are the frequency and wavelength of the wave? Also, what is the wave speed and what is the direction of the wave?

Solution

From the form of the function we can read off the amplitude A, the angular frequency ω and wave number k. Also, the linear density μ is given.

$$A = 0.020\text{ m}\,,\quad \omega = 105\text{ s}^{-1},\quad k = 3.0\text{ m}^{-1}\quad\text{and}\quad \mu = 0.014226\text{ kg}/\text{m}$$

The frequency, wavelength and speed follow from formulas for sinusoidal waves.

$$f = \frac{\omega}{2\pi} = 16.7\text{ Hz},\quad \lambda = \frac{2\pi}{k} = 2.09\text{ m}\quad\text{and}\quad v = \frac{\omega}{k} = 35\,\frac{\text{m}}{\text{s}} = f\,\lambda$$

The solution for a pulse is $f(x \mp v\,t)$, where the negative sign means the pulse is moving in the positive-x direction and positive implies the negative-x direction. Since the relative sign between ωt and $k x$ terms is positive, the wave is moving in the negative-x direction.

(b) What is the maximum speed of a point on the wire as the wave passes.

Solution

A point on a string (or wire) moves in simple harmonic motion as a sinusoidal wave passes. The maximum speed for simple harmonic motion is

$$v_{max} = \omega\,A = 2.1\text{ m}/\text{s}\,.$$

Note that the wave speed is quite distinct from the speed of a point on the wire.

(c) What is the tension in the wire?

Solution

The tension in the wire can be found from the wave speed v and the linear density μ.

$$v = \sqrt{\frac{F_T}{\mu}} \implies F_T = \mu\,v^2 = 17.4\text{ N}$$

(d) The volume density (mass/volume) of steel is $8050\text{ kg}/\text{m}^3$. What is the diameter of the wire?

Solution

The cross-sectional area of the wire can be found from the linear density μ and the volume density $\rho = 8050\ \mathrm{kg}/\mathrm{m}^3$.

$$\mu = \frac{m}{\ell} \implies \rho = \frac{m}{V} = \frac{m}{\text{Area} \times \ell} = \frac{\mu}{\text{Area}} \implies \text{Area} = \frac{\mu}{\rho} = 1.767 \times 10^{-6}\ \mathrm{m}^2$$

$$\text{Area} = \pi \left(\frac{d}{2}\right)^2 \implies d = 2\sqrt{\frac{\text{Area}}{\pi}} = 1.50\ \mathrm{mm}$$

(e) At what rate does energy flow down the wire?

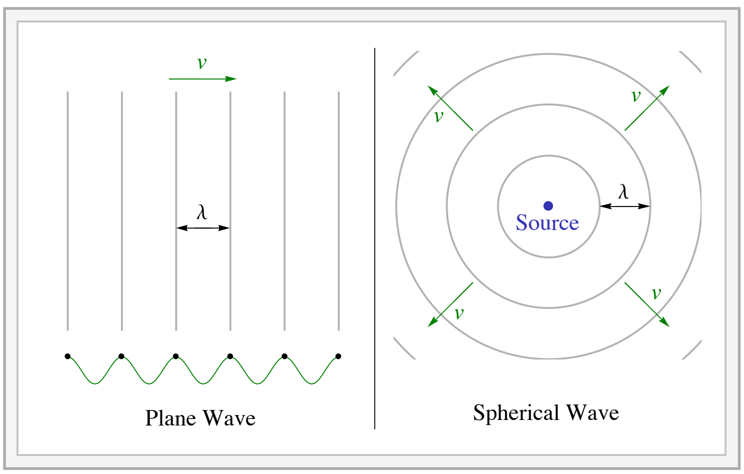
Solution

The rate of energy flow is the power. Note that v is the wave speed.

$$\mathcal{P} = \frac{1}{2}\mu A^2 \omega^2 v = 1.10\ \mathrm{W}$$

Waves in Three Dimensions

Plane Waves and Spherical Waves



Interactive Figure

In three dimensions we define a surface of constant phase, typically the crest of the wave, to be a wave front. An important notion is that of a plane wave; take the wave to move in the positive x-direction with the disturbance u being uniform along the yz-plane. This turns our disturbance from a genuinely three dimensional function $u(x, y, z, t)$ into being a function of only x, spatially.

$$u(x, t) = A\cos(k\,x - \omega\,t + \delta) \quad \text{(Plane Wave)}$$

Wave fronts correspond to planes parallel to the yz-plane, separated by one wavelength and moving in the positive x-direction at speed v.

A point source produces a spherical wave. We can write the disturbance u as a function of r, the distance form the source and time t.

$$u(r, t) = A(r)\cos(k\,r - \omega\,t + \delta) \quad \text{(Spherical Wave)}$$

The wave fronts are concentric spheres separated by λ. The waves fronts move away from the source at the speed v.

Intensity

We saw in the discussion of waves on a string that the average power transmitted down the string was proportional to the wave amplitude squared, $\overline{\mathcal{P}} \propto \text{Amplitude}^2$. For a wave that moves though three dimensions, what is analogous to power in the string case is the intensity, which we define as power per area.

$$I = \frac{\mathcal{P}_{\text{ave}}}{A} = \frac{E}{A\,\Delta t} \quad \left(\text{Intensity} = \frac{\text{Power}}{\text{Area}} = \frac{\text{Average Energy}}{\text{Area} \times \text{time}}\right)$$

The intensity will always be proportional to the amplitude squared.

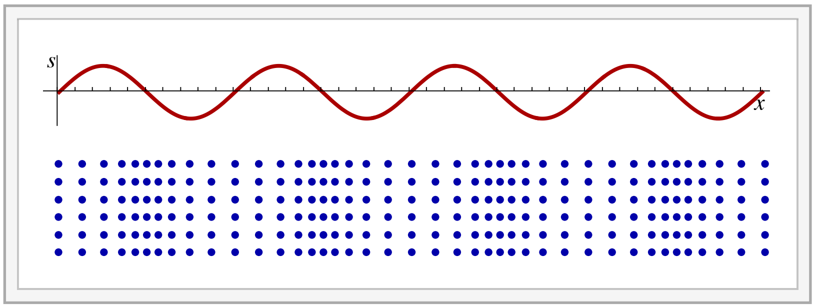
With a plane wave the area stays uniform so the intensity is uniform. For a spherical wave the relevant area is the surface area of a sphere, where $A = 4\,\pi\,r^2$. It follows that the intensity varies with r, the distance form the source, by an inverse square law.

$$I = \frac{\mathcal{P}_{\text{ave}}}{4\,\pi\,r^2}$$

$\overline{\mathcal{P}}$ is the total power output of the source, the rate at which it emits energy.

Sound Waves in a Fluid

Displacement and Pressure

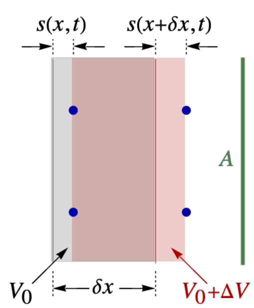


Interactive Figure

We now consider sound waves in a fluid (a liquid or a gas.) Our one-dimensional model consists of a plane wave, which is three dimensional but where nothing varies along the yz-plane, so it is a function of the one spatial variable x. Alternatively, we can describe it as a fluid in a pipe with frictionless walls, making the cross-sectional area A unimportant.

We may view sound waves in a fluid as pressure waves or as displacement waves. Consider the motion of particles (molecules) relative to their equilibrium positions in a fluid. The displacement s is the position of particles relative to their equilibrium position. In our one-dimensional model $s(x, t)$ is the longitudinal displacement of particles as a function of position x and time t. Longitudinal means that the displacement is along the direction of propagation, the $\pm x$ direction. Similarly, we can write the pressure as a function of x and t, $P(x, t)$. By pressure P we mean the *gauge pressure*, which is the difference between P_{absolute}, the absolute pressure, and P_0, the atmospheric pressure or the ambient pressure in the fluid. Although the absolute pressure cannot be negative, the gauge pressure P can.

$$P = P_{\text{gauge}} = P_{\text{absolute}} - P_0$$



To relate the pressure and displacement recall the discussion of elasticity $\Delta P = -B\,\Delta V / V$, where B is the bulk modulus. Consider a plug of fluid of small width δx with cross-sectional area A. ΔP becomes the gauge pressure P. The volume of the plug is $V = A\,\delta x$. The change in the volume of the plug depends on the difference of the displacement values at the endpoints:

$$\begin{aligned}\Delta V &= (V + \Delta V) - V = A\,[\delta x + s(x + \delta x, t) - s(x, t)] - A\,\delta x \\ &= A\,[s(x + \delta x, t) - s(x, t)] = A\,\frac{\partial s}{\partial x}\,\delta x = \frac{\partial s}{\partial x}\,V,\end{aligned}$$

where the last equality uses that the δx is small. It follows then that

$$P = -B \frac{\partial s}{\partial x} .$$

We will refer to this as the pressure-displacement relationship. Note that $\partial s / \partial x$ is the bulk strain and this is also Hooke's law, the stress-strain relationship.

The Wave Equation and the Speed of Sound

Continuing with our one-dimensional model, we will now use Newton's second law and our previous pressure-displacement relationship to derive a one dimensional wave equation. This will mirror our discussion of waves on a string and we will similarly read off the wave speed from the wave equation. Consider a plug of fluid of cross-sectional area A and small width δx. The small net force acting on this small plug is

$$F_{\text{net}} = -A\,[P(x + \delta x, t) - P(x, t)] = -A \frac{\partial P}{\partial x} \delta x ,$$

where we have used that force is pressure times area, $F = P\,A$, and that δx is small. The minus sign follows from pressure creating an inward force on the plug. Using our pressure-displacement relationship this becomes

$$F_{\text{net}} = B \frac{\partial^2 s}{\partial x^2} A\,\delta x .$$

The acceleration of the infinitesimal segment is $\partial^2 s / \partial t^2$ and its small mass is $\rho\,A\,\delta x$, where ρ is the density (mass/volume). Applying the second law gives:

$$F_{\text{net}} = m\,a \implies B \frac{\partial^2 s}{\partial x^2} A\,\delta x = (\rho\,A\,\delta x) \frac{\partial^2 s}{\partial t^2}.$$

We can rewrite this as

$$\frac{\partial^2 s}{\partial x^2} = \frac{1}{B / \rho} \frac{\partial^2 s}{\partial t^2}.$$

If we compare this with the one dimensional wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{v^2} \frac{\partial^2 u}{\partial t^2},$$

we can see that with $u = s$, the disturbance is the longitudinal displacement, we get the wave equation with the speed given by

$$v = \sqrt{\frac{B}{\rho}} .$$

Note that if we take the x partial derivative of the wave equation with displacement s and use $P = -B\,\partial s / \partial x$ we get a wave equation with pressure as the disturbance.

$$\frac{\partial^2 P}{\partial x^2} = \frac{1}{B / \rho} \frac{\partial^2 P}{\partial t^2}$$

Now that we have found a wave equation for sound in a fluid we can apply the discussion of solutions to the wave equation to sound waves. The simplest waveforms are sinusoidal. There are two notions of amplitude, the displacement amplitude s_{max} and the pressure amplitude P_{max}. Using s for the disturbance u and s_{max} as the amplitude A we get:

$$s(x, t) = s_{\text{max}} \cos(k\,x \mp \omega\,t + \delta).$$

Using the pressure-displacement relationship $P = -B\,\partial s / \partial x$ we get sinusoidal pressure waves.

$$\begin{aligned} P(x, t) &= -B \frac{\partial}{\partial x} s(x, t) = B\,k\,s_{\text{max}} \sin(k\,x \mp \omega\,t + \delta) \\ &= P_{\text{max}} \sin(k\,x \mp \omega\,t + \delta) \end{aligned}$$

From this we can read off the pressure amplitude P_{max}.

$$P_{\text{max}} = B\,k\,s_{\text{max}} = \rho\,v\,\omega\,s_{\text{max}}$$

The second equality above follows from $B = \rho\,v^2$ and $\omega = k\,v$.

Example L.4 - Bulk Modulus of Air

At 20 °C and 1 atm, the speed of sound in air is 343 m/s and the density of air is 1.203 kg/m^3. Using these values, what is the bulk modulus of air.

Solution

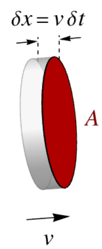
$$v = 343\,\frac{\text{m}}{\text{s}},\quad \rho = 1.203\,\frac{\text{kg}}{\text{m}^3} \text{ and } v = \sqrt{\frac{B}{\rho}} \implies B = \rho\, v^2 = 142\,000\text{ Pa}$$

Intensity and Sound Level

For waves in three dimensions we describe the flow of energy with the intensity, defined as average power per area.

$$I = \frac{\mathcal{P}_{\text{ave}}}{A} = \frac{E_{\text{ave}}}{A\,\Delta t} \quad \left(\text{Intensity} = \frac{\text{Power}}{\text{Area}} = \frac{\text{Average Energy}}{\text{Area} \times \text{time}}\right)$$

Consider a cross-section of area A in the yz-plane. With the wave moving in the positive x-direction then if we extend the cross-section in the negative x-direction by $\delta x = v\,\delta t$ then this forms a right cylinder, a plug of fluid that passes the cross-section in the time δt.



The mass of the plug is

$$m = \rho\, A\, v\, \delta t\,.$$

Choosing the δt to be small then we have a small mass moving in simple harmonic motion. The average energy of the vibrating plug is

$$E_{\text{ave}} = \frac{1}{2}\, m\, v_{\text{max}}^2 = \frac{1}{2}\, \rho\, A\, v\, \delta t\, \omega^2\, s_{\text{max}}^2\,.$$

With $I = \frac{\overline{\mathcal{P}}}{A} = \frac{\overline{E}}{A\,\delta t}$ we can find the intensity:

$$I = \frac{1}{2}\, \rho\, v\; \omega^2\, s_{\text{max}}^2\,.$$

To write the intensity in terms of the pressure amplitude use $P_{\text{max}} = \rho\, v\, \omega\, s_{\text{max}}$.

$$I = \frac{P_{\text{max}}^2}{2\,\rho\, v}$$

We can perceive sound over a wide range of intensities; this makes a logarithmic scale convenient for measuring loudness. We will call our measure of loudness β, the *sound level*, and measure it in decibels dB. If we define

$$I_0 = 10^{-12}\,\frac{\text{W}}{\text{m}^2}$$

as the *threshold of hearing* then we can define β, in dB, by:

$$\beta = (10\text{ dB})\log\left(\frac{I}{I_0}\right).$$

Example L.5 - Quiet Please!

For sound (in air) at a frequency of 2000 Hz the threshold of pain is a sound level of about 115 dB. At this sound level:

(a) What is the intensity of the sound?

Solution

$$I_0 = 10^{-12}\ \frac{\text{W}}{\text{m}^2} \text{ and } \beta = (10\text{ dB})\log\left(\frac{I}{I_0}\right) = 115\text{ dB} \Longrightarrow I = I_0\ 10^{\beta/(10\text{ dB})} = 0.316\ \frac{\text{W}}{\text{m}^2}$$

(b) What is the pressure amplitude of the wave?

Solution

$$v = 343\ \frac{\text{m}}{\text{s}},\ \rho = 1.203\ \frac{\text{kg}}{\text{m}^3} \text{ and } I = \frac{P_{max}^2}{2\,\rho\,v} \Longrightarrow P_{max} = \sqrt{2\,\rho\,v\,I} = 16.15\text{ Pa}$$

(c) What is the displacement amplitude?

Solution

$$P_{max} = \rho\,v\,\omega\,s_{max}$$

We are given the frequency and that gives ω.

$$f = 2000\text{ Hz} \Longrightarrow \omega = 2\,\pi\,f = 12\,570\text{ s}^{-1}$$

We can then solve for the displacement amplitude.

$$s_{max} = \frac{P_{max}}{\rho\,v\,\omega} = 3.12\times 10^{-6}\text{ m}$$

Sound Waves in Solids

There is no shear stress in a fluid so we cannot have transverse sound waves in a fluid. In a solid there is shear and we get transverse sound waves in addition to the longitudinal compression waves. These two types of waves travel at different speeds. For the case of sound waves in a fluid the pressure-displacement relationship was our stress-strain proportionality. Playing the analogous role of pressure we have the tensile stress and shear stress which we will refer to as $P_{\text{longitudinal}}$ and $P_{\text{transverse}}$. The tensile strain and shear strain are $\partial s/\partial x$ and $\partial y/\partial x$, where the longitudinal displacement is s as with the case of fluids and the transverse displacement is y, as with waves on a string. Young's modulus and the shear modulus give us the stress-strain proportionality.

$$P_{\text{longitudinal}} = Y\,\frac{\partial s}{\partial x} \text{ and } P_{\text{transverse}} = S\,\frac{\partial y}{\partial x}\,.$$

The net force on a small slab between x and $x + \delta x$ is

$$F_{\text{net,longitudinal}} = A\,\frac{\partial P_{\text{longitudinal}}}{\partial x}\,\delta x$$
$$\text{and } F_{\text{net,transverse}} = A\,\frac{\partial P_{\text{transverse}}}{\partial x}\,\delta x$$

and the mass of the small slab is $\rho\,A\,\delta x$. The accelerations are

$$a_{\text{longitudinal}} = \frac{\partial^2 s}{\partial t^2} \text{ and } a_{\text{transverse}} = \frac{\partial^2 y}{\partial t^2}.$$

Applying the second law, $F_{\text{net}} = m\,a$, to both longitudinal and transverse cases gives our wave equations.

$$Y\,\frac{\partial^2 s}{\partial x^2} = \rho\,\frac{\partial^2 s}{\partial t^2} \text{ and } S\,\frac{\partial^2 y}{\partial x^2} = \rho\,\frac{\partial^2 y}{\partial t^2}$$

From this we can read off the waves speeds for both types of sound waves.

$$v_{\text{longitudinal}} = \sqrt{\frac{Y}{\rho}} \text{ and } v_{\text{transverse}} = \sqrt{\frac{S}{\rho}}$$

In geology these two waves speeds are significant when studying seismology. The longitudinal waves are called p-waves, the primary waves, and the transverse are called s-waves, the secondary waves. A seismograph observes the first s-waves arriving later than the first p-waves; the time difference, combined with knowledge of the wave speeds, can give the distance to the epicenter of an earthquake and the time of it. The earth's core is molten; since transverse waves cannot propagate through a liquid, the s-waves have a limited range. The p-waves are seen world-wide.