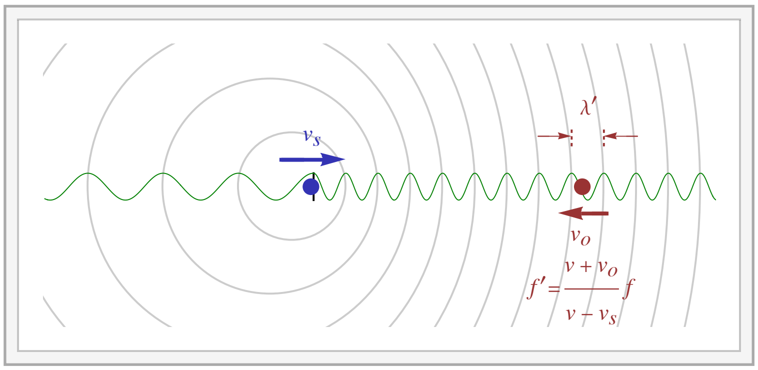
Chapter M

Applications of Waves

Blinn College - Physics 2325 - Terry Honan

The Doppler Effect and Shock Waves

When a train is moving toward you with its whistle blowing the observed pitch of the whistle, its frequency, increases. When moving away its pitch decreases. This is known as the Doppler effect; it is a general property of waves but we will discuss it in the context of sound waves. Take the speed of sound to be v; this is the speed of sound with respect to the stationary atmosphere. Consider v_s and v_o to be the approaching velocities of the source, producing the sound, and the observer, listening to it. By "approaching velocities" we mean we will take their signs to be positive when moving toward the other. If the source or observer is moving away from the other, we will take the sign of its velocity to be negative.



Interactive Figure

λ' is the wavelength as heard by the observer. The wavelength at the observer λ' is the distance between wavefronts; it is smaller by $v_s\,T$, where $T = 1/f$, since the later wavefront left the source after it had moved that distance. The wave moves at speed v relative to the stationary atmosphere, so relative to the moving observer the wave speed is $v + v_o$.

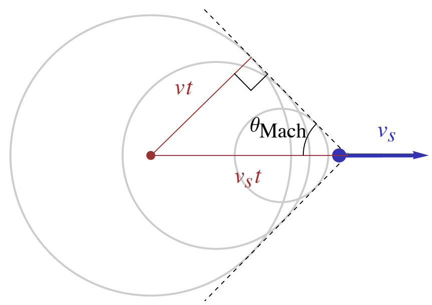
$$\lambda' = \lambda - v_s\,T = \frac{v}{f} - \frac{v_s}{f} = \frac{v - v_s}{f} \text{ and } v' = v + v_o$$

Combining these two expressions we can find the wave frequency as heard by the observer.

$$f' = \frac{v'}{\lambda'} = \frac{v + v_o}{v - v_s}\,f$$

When the source travels faster than the speed of sound, the wave fronts meet to form a conical shock wave trailing the source. This shock wave is the sonic boom from a hypersonic jet. Another example is the wedge-shaped wake behind a boat; the boat is moving faster than the surface water waves to create a shock wave. The angle of the shock wave behind the source is called the Mach angle, θ_{Mach}. In a time t the spherical wave front moves by $v\,t$ and the source moves by $v_s\,t$; the Mach angle is found by simple trigonometry.

$$\sin\theta_{\text{Mach}} = \frac{v}{v_s}$$



Superposition

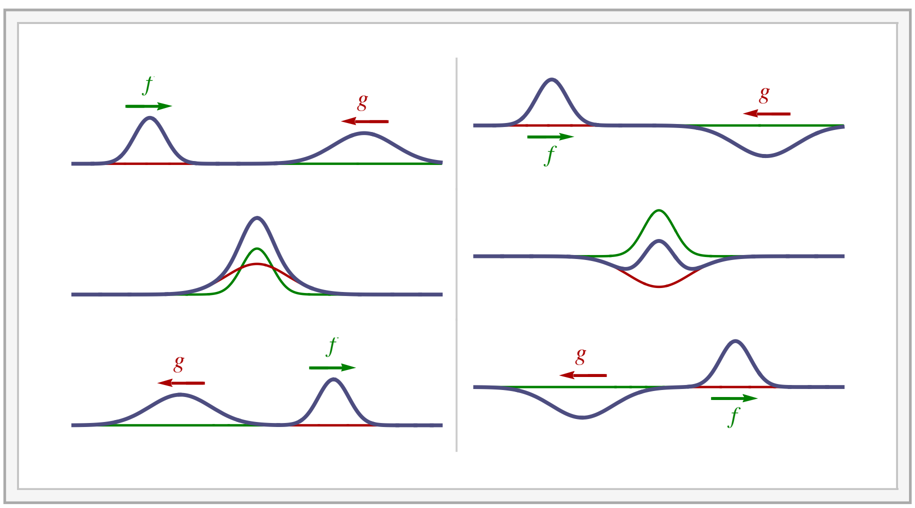
Superposition in One Dimension

For any wave equation, including generalized dispersive equations, we have the principle of superposition. This means that if u_1 and u_2 are wave solutions then their sum $a\,u_1 + b\,u_2$ is a solution, where a and b are constants. For the one-dimensional wave equation this is easy to show. If $u_1(x, t)$ and $u_2(x, t)$ solve the one-dimensional wave equation then $a\,u_1(x, t) + b\,u_2(x, t)$ must also solve it. This follows from the linearity of derivatives; constants come out of derivatives and the derivative of the sum of functions is the sum of their derivatives.

In one dimension the only way that we can combine different waves is if they are traveling in opposite directions since the general solution to the wave equation is

$$u(x, t) = f(x - v\,t) + g(x + v\,t)$$

and two pulses moving in the same direction move at the same speed. Suppose the f and g solutions are pulses moving toward each other. Our general solution shows that when a pulse moving in the positive x-direction, $f(x - v\,t)$, meets a pulse moving in the opposite direction, $g(x + v\,t)$, they add where they overlap and move off unchanged by the interaction.



Interactive Figure

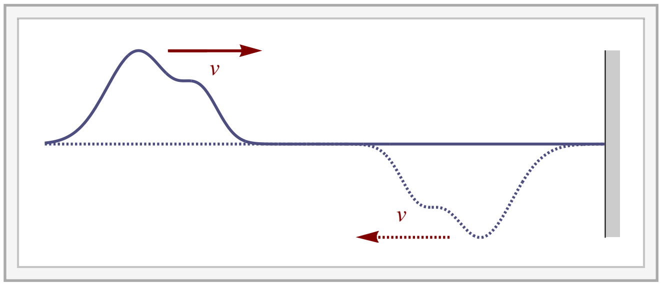
Boundary Conditions: Reflection and Transmission

Reflection from a Fixed Point

Suppose a pulse on a string hits a point where the string is held fixed at the equilibrium position, where $y = 0$. The result is there will be an inverted pulse reflecting backward. To make a general statement about waves we will use u for the disturbance instead of y. If we take the fixed position to be at $x = 0$, then the boundary condition becomes

$$u(0, t) = 0\,.$$

The boundary condition forces the pulse to become inverted on reflection.



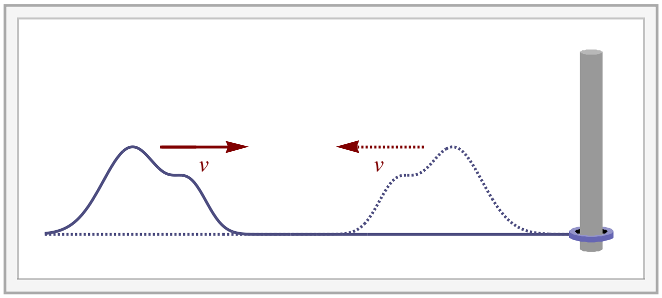
Interactive Figure

Reflection from a Free End with Tension

Now imagine a pulse traveling down a string toward a ring sliding frictionlessly on rod. The rod provides a tension to the string but gives no transverse (perpendicular) force. Since the transverse force is $T_\perp = T\,\partial y/\partial x$ our boundary condition is that the x partial derivative of the disturbance y vanishes. Using u as our generic disturbance then our wave boundary condition becomes

$$\frac{\partial u}{\partial x}(0,\, t) = 0\,,$$

where we have again chosen the boundary to be at $x = 0$. The effect of this is the pulse reflects but is not inverted.



Interactive Figure

Connected Strings

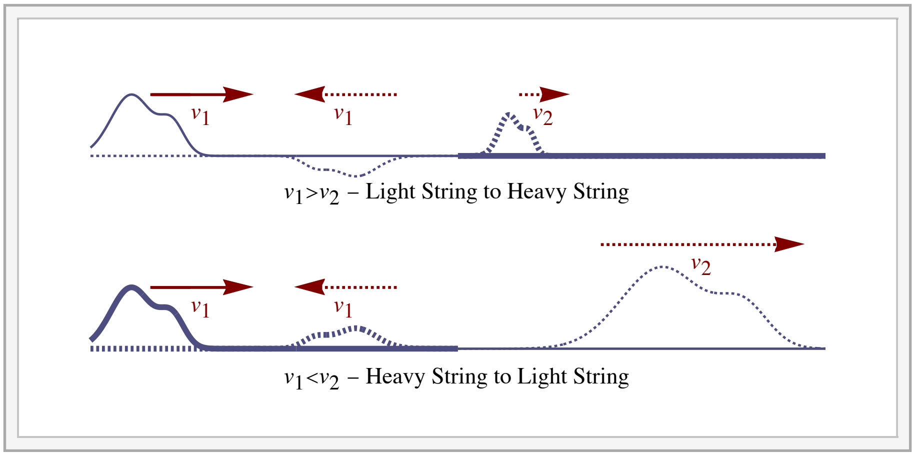
Now imagine two strings with different linear densities μ tied together. What happens to a pulse when it moves from one string to another? Since the speed of waves is $v = \sqrt{T/\mu}$ and the tension T must be the same on both sides the side with the heavier rope (larger μ) has the smaller speed. When we pose our discussion in terms of speeds then we are making general statements about one dimensional waves and not specialized toward the case of waves on strings.

Take the boundary between the strings to be at $x = 0$. Our boundary conditions become

$$u(0^-,\, t) = u\big(0^+,\, t\big) \;\text{ and }\; \frac{\partial u}{\partial x}(0^-,\, t) = \frac{\partial u}{\partial x}\big(0^+,\, t\big)\,;$$

that is: the disturbance and its x-derivative must be continuous. Note that the 0^- and 0^+ notation refers to the left- and right-handed limits at 0, $f\big(0^\pm\big) = \lim_{x\to 0^\pm} f(x)$. For waves on a string, the continuity of u means simply that the two strings are connected and the continuity of the derivative follows from Newton's third law applied to the transverse force.

When a pulse travels down a string toward a boundary with a string of different linear density there are three pulses to describe. The original pulse moving forward toward the boundary is called the *incident pulse*. Moving forward away from the boundary on the other side is the *transmitted pulse*. Moving backward on the original side of the boundary is the *reflected pulse*. The following figure shows the incident, reflected and transmitted pulses for difference relative values of v_1 and v_2. The figure reflects the proper solution to the one-dimensional wave equation with the above boundary conditions using the appropriate speeds.

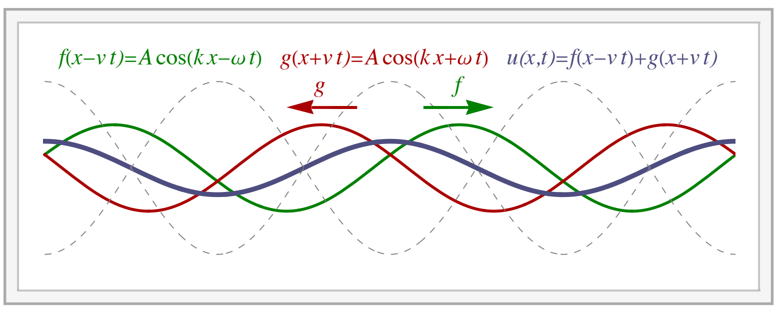


Interactive Figure

Standing Waves

Standing Waves

When a mechanical system, say a mass-spring system, is driven by a periodic force it will oscillate at the driving frequency. An extended system, like a building, a bridge or a musical instrument will have many natural frequencies of vibration. We will now study these natural frequencies for several simple cases that involve standing waves.



Interactive Figure

To describe a standing wave consider a one-dimensional case with two sinusoidal waves with the same amplitude and frequency but moving in opposite directions.

$$u(x,\,t) = A\cos(kx - \omega t) + A\cos(kx + \omega t)$$

To add these two waves we will use trig. identities. Start with the formulas for cosine of the sum and difference of two angles and add the results.

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta \quad \Longrightarrow \quad \cos(\alpha+\beta) + \cos(\alpha-\beta) = 2\cos\alpha\cos\beta.$$

Using $\alpha = k\,x$ and $\beta = \omega\,t$ we get our result.

$$u(x,\,t) = 2\,A\cos(kx)\cos(\omega t)$$

The positions where the disturbance u is zero at all times are called *nodes*. We refer to the positions where the disturbance has its largest amplitude oscillation as *antinodes*.

$$\cos(kx) = 0 \quad \Longleftrightarrow \quad \text{nodes}$$

$$\cos(kx) = \pm 1 \quad \Longleftrightarrow \quad \text{antinodes}$$

Note that the distance between nodes is half a wavelength. The wavelength and frequency are the same as for the two left and right moving sinusoidal waves, so it follows the we still have $f\,\lambda = v$.

Standing Waves on a String

Suppose a string under tension is fixed at either end. If you pluck this string it will vibrate in a fairly complex pattern. This complex pattern can be understood in terms of linear superpositions of some simple vibrational modes, called harmonics. We will now describe these harmonics.

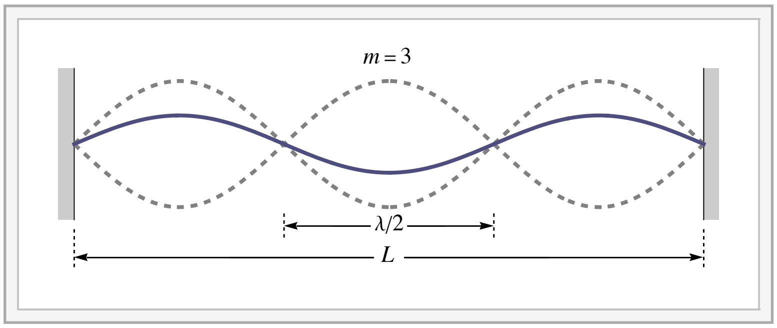
The string has tension T and linear density μ so the wave speed is $v = \sqrt{T/\mu}$. It is fixed at $x = 0$ and $x = L$. We insist on a standing wave solution that is zero at $x = 0$; to do this shift the spatial part of the trig. function to $\sin(k\,x)$, since $\sin 0 = 0$. Recall that $u = y$ for a string. A general expression for the standing wave becomes:

$$y(x, t) = y_{\max} \sin(k\,x) \cos(\omega\,t + \phi)$$

where $y_{\max}$ is the amplitude of the resulting standing waves and not the amplitudes of the left-right moving waves in the previous subsection. For the solution to be zero at L we have $\sin(k\,L) = 0$. This implies $k\,L = m\,\pi$. Since $k = 2\,\pi/\lambda$ we can find the wavelengths of our harmonic modes.

$$\lambda_m = \frac{2\,L}{m} \quad \text{where } m = 1, 2, 3, \ ...$$

Let us get at this expression above in a less mathematical way. The distance between nodes of the resulting standing wave pattern is half a wavelength. m of these half-wavelengths fit into L, so $m\,\lambda/2 = L$.



Interactive Figure

The frequency follows from $f\,\lambda = v$.

$$f_m = m\,f_1 \quad \text{where} \quad f_1 = \frac{v}{2\,L}$$

f_1 is the fundamental frequency or the first harmonic. The higher m values are the higher harmonics. An instrument will always produce higher harmonics, integer multiples of the fundamentals, but the frequencies we associate with an instrument are the fundamental frequencies. We tune to correct the fundamental frequencies. When we tune a stringed instrument we vary the tension in the string, thus varying the wave speed; given the string's length, we have a unique fundamental frequency for a string. A piano has a separate string for each note, thus tuning a piano is a very tedious procedure. A standard guitar has just six strings but the player's fingers moving on the frets creates many effective lengths for each string, producing many different notes.

Standing Sound Waves in a Pipe

We will now consider standing waves formed by sound in a pipe. The boundary conditions at the ends of the pipe depend on whether the end is open or closed. At an open end the pressure P is fixed at zero at the end, since the pressure we use in our wave discussion is the gauge pressure which is the pressure difference from that outside the pipe. The displacement s at the end of the pipe is unconstrained. The result is at an open end we have a pressure node and a displacement antinode.

$$\text{Open end at } x_0: \ s(x_0, t) = \pm s_{\max} \text{ (antinode)}, \ \ P(x_0, t) = 0 \ \text{(node)}$$

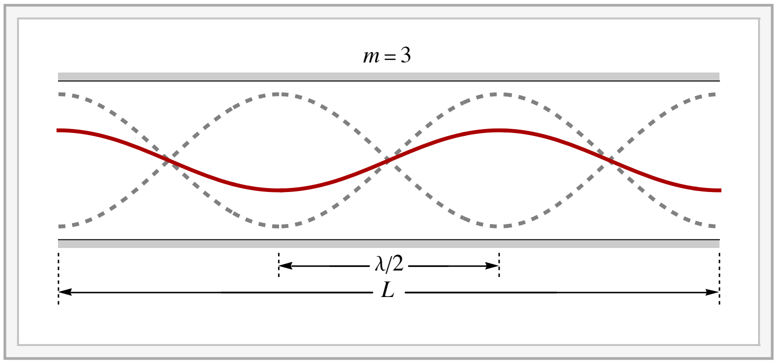
At a closed end of a pipe the displacement is forced to be zero, a node and the pressure is unconstrained, an antinode.

$$\text{Closed end at } x_0: \ s(x_0, t) = 0 \text{ (node)}, \ \ P(x_0, t) = \pm P_{\max} \ \text{(antinode)}$$

The pressure-displacement relationship, $P = -B\,\partial s/\partial x$, allows us to see this differently; taking derivatives of trig. functions swaps sine and cosine and thus swaps nodes and antinodes.

Now consider a pipe of length L with both ends open. From the pressure perspective there is a node at either end, just as the case of a string with both ends fixed. The displacement perspective looks different but the counting is still the same; we need an integer number of half wavelengths over the length L.

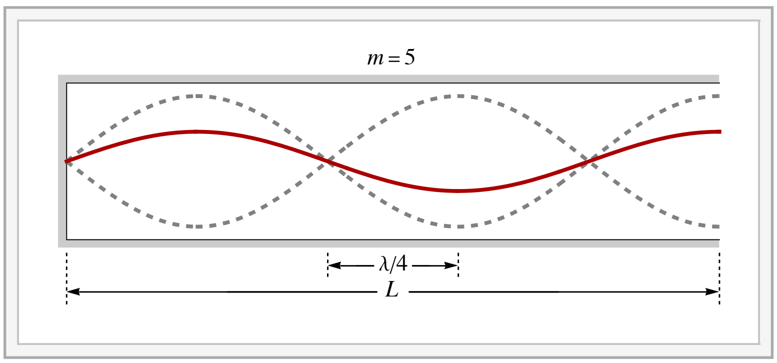
$$\lambda_m = \frac{2\,L}{m} \ \text{and} \ f_m = m\,f_1 \ \text{with} \ f_1 = \frac{v}{2\,L} \quad (m = 1, 2, 3, \ ...)$$



Interactive Figure

Now we turn to a pipe of length L with one end open and the other closed. From the pressure perspective there is a node at the open end and an antinode at the closed end. The displacement point of view gives the reverse, an antinode at the open end and a node at the closed end. To get the standing waves to fit these boundary conditions we must have an odd number of quarter wavelengths spanning the length L, so $L = m\,\lambda/4$ where m is odd.

$$\lambda_m = \frac{4\,L}{m} \text{ and } f_m = m\,f_1 \text{ with } f_1 = \frac{v}{4\,L} \quad (m = 1, 3, 5, \ ...)$$



Interactive Figure

Beats

Consider interference between two sound waves that differ slightly in frequency. The resulting sound is heard to pulse at a frequency much smaller than that of the component waves. At some position from some source a wave will have the form: $u(t) = A\cos(\omega\,t + \phi)$. To make the result its most dramatic we will assume that at the position where the combined waves are heard, the amplitudes of the two waves are the same. Also, for simplicity, we will choose $t = 0$ to be when both waves are at a phase $\phi = 0$. The two waves become:

$$u_1(t) = A\cos(\omega_1\,t) \text{ and } u_2(t) = A\cos(\omega_2\,t).$$

To add these two waves we will use the same trig. identity we used with standing waves.

$$\cos(\alpha + \beta) + \cos(\alpha - \beta) = 2\cos\alpha\cos\beta.$$

Using $\alpha = \frac{1}{2}(\omega_1 + \omega_2)\,t$ and $\beta = \frac{1}{2}(\omega_1 - \omega_2)\,t$ we get:

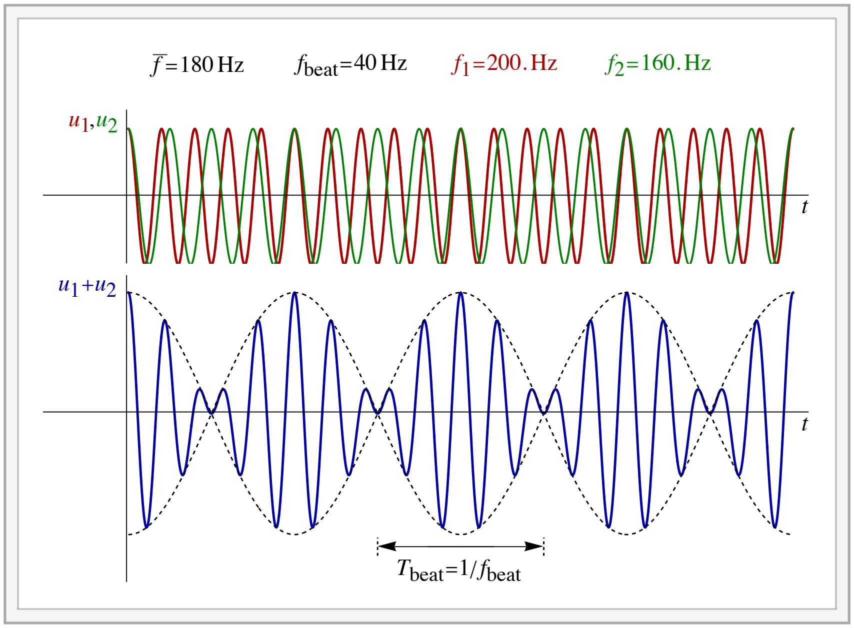
$$\begin{aligned} u(t) &= u_1(t) + u_2(t) = A\cos(\omega_1\,t) + A\cos(\omega_2\,t) \\ &= 2\,A\cos\left(\frac{\omega_1 + \omega_2}{2}\,t\right)\cos\left(\frac{\omega_1 - \omega_2}{2}\,t\right). \end{aligned}$$

This can be written in terms of the average (angular) frequency ϖ and the beat (angular) frequency ω_{beat}.

$$u(t) = 2\,A\cos(\varpi\,t)\cos\left(\frac{1}{2}\,\omega_{\text{beat}}\,t\right) \quad \text{where } \varpi = \frac{\omega_1 + \omega_2}{2} \text{ and } \omega_{\text{beat}} = |\omega_1 - \omega_2|$$

Using $\omega = 2\,\pi\,f$ we get the average frequency and beat frequency:

$$u(t) = 2\,A\cos\left(2\,\pi\,\overline{f}\,t\right)\cos(\pi\,f_{\text{beat}}\,t) \quad \text{where } \overline{f} = \frac{f_1 + f_2}{2} \text{ and } f_{\text{beat}} = |f_1 - f_2|$$



Interactive Figure

The frequency $\overline{f}$ oscillates rapidly. Since the function $\cos(2\pi \overline{f}\, t)$ varies between ± 1, the function $u(t)$ stays between $\pm 2A\cos(\pi f_{beat}\, t)$, this is called the envelope function,