

Chapter A

Electric Charges and Electric Fields

Blinn College - Physics 2326 - Terry Honan

A.1 - Electric Charge

Properties of Charge

There are two types of electric charge. Like charges repel and unlike attract.

We can label the types with signs. The choice of positive and negative is arbitrary.

Electric charge is conserved.

The other fundamentally conserved quantities are: energy, linear momentum and angular momentum; these were discussed in the first semester mechanics course. A conserved quantity cannot be created or destroyed. To charge a body one can add charge to it or remove charge from it.

Electric charge is quantized.

Any charge is an integer multiple of the elementary charge e . The unit of charge is C = coulombs, which is considered a fundamental unit in the SI system.

$$Q = n e, \text{ where } e = 1.602 \times 10^{-19} \text{ C and } n \text{ is an integer}$$

The historical definition of the Coulomb will be given later, when we discuss the magnetic force between two conductors. Since the May 2019 redefinition of the SI system, the coulomb is defined by setting the value for the elementary charge e to be exactly $1.60217634 \times 10^{-19} \text{ C}$.

Normal matter consists of protons, neutrons and electrons with charges shown here.

Particle	Charge
proton	$+e$
neutron	0
electron	$-e$

It is clear that any combination of these will satisfy the charge quantization rule $Q = n e$, where $n = 0, \pm 1, \pm 2, \pm 3, \dots$

Units: The SI unit for charge is: C

Aside on Quarks and Particles

Fractionally Charged Quarks

Quarks are particles with fractional charge that combine to form a class of particles called hadrons, which includes protons and neutrons. Normal matter consists of combinations of two quarks: the *up* quark u and the *down* quark d , and electrons. The electron is not made up of quarks and is believed to be fundamental.

Quark	Charge
u	$+\frac{2}{3} e$
d	$-\frac{1}{3} e$

Although, it appears that the existence of fractionally charge quarks violates the charge quantization condition, fractionally charged quarks only occur in integer-charged combinations in nature. Combinations with three quarks form a class of particles called baryons, with neutrons and protons as examples. In addition to quarks there are antiquarks, which are the antiparticle counterpart of quarks; antiparticles are particles of the same mass but have the opposite charge. Three antiquarks make antibaryons. Also, a quark q and an antiquark \bar{q} combine to form a class of particles called mesons.

Particle	Quark Content
proton	$u u d$

neutron	$u d d$
antiproton	$\bar{u} \bar{u} \bar{d}$
antineutron	$\bar{u} \bar{d} \bar{d}$
positive pion	$u \bar{d}$
negative pion	$\bar{u} d$

In addition to the up and down quarks there are four more quarks: the strange quark s , the charmed quark c , the bottom quark b and the top quark t . The charmed and top have the same charge as the up quark and the strange and bottom have the same charge as the down. These exotic quarks do not occur in usual matter but may be created by sufficiently powerful accelerators.

Fundamental Forces and Elementary Particles

The fundamental forces are: electromagnetism, the weak nuclear force, the strong nuclear force and gravity. Elementary particles break up into two categories: fermions which satisfy the Pauli Exclusion principle and bosons which do not. The fermions are the particles of matter and the bosons are the particles of interaction. The current working theory of particle physics, which has remained intact since the mid-1970s, is known as the *Standard model*. This encapsulates the fundamental elementary particles, which are described below, and their forces of interaction. The standard model unifies the weak and electromagnetic forces into what is called the electroweak force. It also describes the strong nuclear force as a separate interaction, but it does not include gravity, which is described by general relativity. The standard model is just provisional, until we can find a better theory; this would need to incorporate gravity, explain dark energy, dark matter and many other mysteries.

Fundamental fermions consist of quarks and leptons. They break up into three families, where the masses get increasingly large with the later families. Each fundamental fermion has an corresponding antiparticle.

	First Family	Second Family	Third Family
Quarks	u (up quark)	c (charm quark)	t (top quark)
	d (down quark)	s (strange quark)	b (bottom quark)
Leptons	ν_e (electron neutrino)	ν_μ (muon neutrino)	ν_τ (tau neutrino)
	e^- (electron)	μ^- (muon)	τ^- (tau)

Fundamental Fermions - the Particles of Matter

It is still a mystery why there are three families.

In our quantum description of interaction, all forces are associated with particles. These forces and their particles are listed below.

Fundamental Interaction		Particle
Standard Model	Electroweak Force	Electromagnetism γ (photon)
		Weak Nuclear Force W^+ , W^- , Z^0 (W and Z bosons)
	Strong Nuclear Force g (eight gluons)	
Gravity		gravitons

Fundamental Forces and their Bosons

All of these particles have been experimentally observed, except the graviton. Only recently, in 2016, have we observed gravity waves; this is a very weak and subtle thing to detect. The graviton is the quantum of this gravitational wave and we are nowhere near having the capability to see individual gravitons. In the standard model all particles with mass, the W and Z bosons and the quarks and leptons, get their mass through what is known as the Higgs mechanism. In the simplest implementation of the Higgs mechanism there is a single residual observable particle, called the Higgs particle or Higgs boson; this particle was observed in 2012.

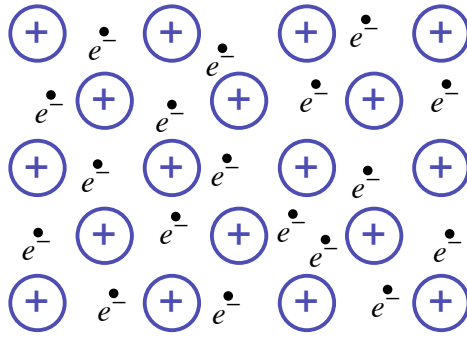
Generation of Particle Masses
H^0 (Higgs boson)

Particles get their mass through the Higgs mechanism, which produces the Higgs boson, as a by-product.

A.2 - Conductors and Insulators

Inside a conductor there are freely moving charges. In an insulator there are not. The most typical conductors are metals, where the charge carriers are electrons. Most of the electrons are tied to their atoms but a small number of electrons from each atom are shared by all atoms in a

sea of electrons, which are free to move. This leaves a background of positively charged atoms with the electrons flowing freely between the atoms.



A metal has a positively charged background of atoms with some electrons e^- removed. These removed electrons are shared in a "sea of electrons" that are free to move.

In semiconductors, the conduction mechanism is different. The charge carriers are either electrons jumping between atomic sites or positively charged holes (A hole is lack of an electron.) jumping between atoms; they are called n-type and p-type semiconductors for negative (electrons) and positive (holes) charge carriers.

When a charge is placed near a conductor, the freely moving charges in the conductor will rearrange themselves. They will move until they reach electrostatic equilibrium.

In an insulator all electrons are either tied to their atom or to bonds between atoms and are not free to move. Whenever the positions of all charges in some distribution is specified, then that must be an insulator. For a conductor we will specify the charge on the conductor but that charge will change positions until reaching equilibrium.

A.3 - Coulomb's Law

The first of the fundamental forces to be understood was gravity; this was described by Newton with his law of universal gravitation. Coulomb successfully described the electrostatic force between charges by analogy to gravity. We will first review Newtonian gravity before discussing Coulomb's law.

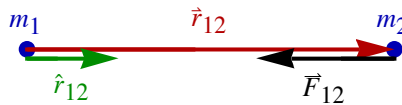
Review of Gravity

Newton's law of gravity is an inverse square law between point masses. If m_1 and m_2 are point masses separated by distance r the magnitude of the force between them is

$$F = G \frac{m_1 m_2}{r^2}.$$

This can be written as a vector expression. Let \vec{r}_{12} be the vector from mass 1 to mass 2 and let \vec{F}_{12} be the force on mass 2 due to mass 1.

$$\begin{aligned}\vec{F}_{12} &= -G \frac{m_1 m_2}{r_{12}^2} \hat{r}_{12} \\ &= -G \frac{m_1 m_2}{r_{12}^3} \vec{r}_{12}\end{aligned}$$



Note that a unit vector is denoted by a " $\hat{}$ " placed over the symbol and any other vector has a " $\vec{}$ " over it. A vector \vec{A} divided by its magnitude $A = \|\vec{A}\|$ is a unit vector $\hat{A} = \frac{\vec{A}}{A}$. So it follows that the unit radial vector \hat{r} is the position vector divided by r , its magnitude. The two vector expressions above are equivalent because

$$\hat{r} = \frac{\vec{r}}{r} \implies \frac{\hat{r}}{r^2} = \frac{\vec{r}}{r^3}$$

Coulomb's Law

Coulomb found the force law for electrostatics by analogy to gravity. Mass is the gravitational analog of charge. There are two key

differences between the electric and gravitational cases. Electric charge can be positive or negative but mass is always positive. The force between two masses is attractive but the force between like charges is repulsive. The electric force is an inverse square law between point charges. The magnitude of the force is

$$F = k_e \frac{|Q_1||Q_2|}{r^2},$$

where the absolute values guarantee a positive result. The constant k_e is a universal constant, like Newton's gravitational constant G . It is related to another constant ϵ_0 , which is usually taken as more fundamental.

$$\epsilon_0 = 8.85 \times 10^{-12} \frac{\text{C}^2}{\text{N} \cdot \text{m}^2}$$

$$k_e \equiv \frac{1}{4\pi\epsilon_0} = 8.99 \times 10^9 \frac{\text{N} \cdot \text{m}^2}{\text{C}^2}$$

Example A.1 - Gravitational versus Electrical Attraction for a Hydrogen Atom

In addition to the attractive electric force between a proton and an electron there is also gravitational attraction as well. Is it important to consider gravity when studying the physics of the hydrogen atom?

What is the ratio of the gravitational to electric attraction between a proton and an electron?

Solution

The values of the relevant constants are :

$$k_e = 8.99 \times 10^9 \text{ N} \cdot \text{m}^2 / \text{C}^2$$

$$G = 6.673 \times 10^{-11} \text{ N} \cdot \text{m}^2 / \text{kg}^2$$

$$e = 1.602 \times 10^{-19} \text{ C}$$

$$m_{\text{electron}} = 9.11 \times 10^{-31} \text{ kg}$$

$$m_{\text{proton}} = 1.673 \times 10^{-27} \text{ kg}$$

The electric and gravitational forces have magnitudes :

$$F_{\text{elec}} = k_e \frac{|Q_1| \cdot |Q_2|}{r^2} = k_e e^2 / r^2$$

$$F_{\text{grav}} = G \frac{m_1 m_2}{r^2} = G m_{\text{electron}} m_{\text{proton}} / r^2$$

We now take the ratio. Note that because both forces vary as inverse square laws, the ratio does not depend on the distance between particles, r .

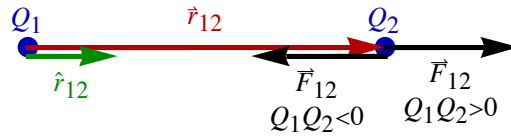
$$\frac{F_{\text{grav}}}{F_{\text{elec}}} = \frac{G m_{\text{electron}} m_{\text{proton}} / r^2}{k_e e^2 / r^2} = \frac{G m_{\text{electron}} m_{\text{proton}}}{k_e e^2} = 4.41 \times 10^{-40}$$

The above number is dimensionless and is thus independent of units. This small numeric value shows that unless 40 digit accuracy is needed, we can ignore gravity when studying the hydrogen atom. The smallness of this value poses a fundamental question: How does some underlying theory that unifies gravity with the other forces, the strong nuclear force and the electroweak force, give rise to such a small dimensionless number. There is no such unified theory now but it is considered an ultimate goal of physics.

To get a vector expression for the force we must include both possibilities, attractive and repulsive. In the case of like charges $Q_1 Q_2 > 0$ and unlike charges give $Q_1 Q_2 < 0$.

$$\vec{F}_{12} = k_e \frac{Q_1 Q_2}{r_{12}^2} \hat{r}_{12}$$

$$= k_e \frac{Q_1 Q_2}{r_{12}^3} \vec{r}_{12}$$

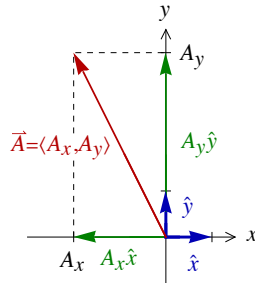


Note that the above expressions differ from the gravitational result by a sign. This is because in the gravitational case there are only like "charges" (mass is positive) and it is attractive but in the electric case, like charges repel.

To find the force on a charge due to a distribution of charge one adds the forces due to each charge in the distribution. Force is a vector and this addition is then vector addition.

Vector Notation

Vector variables will be shown with an arrow "→" over its top. A 2-dimensional vector \vec{A} will be written a pair of real numbers A_x and A_y called components. (A 3D vector is a triple.) We will use the "angled-bracket" $\vec{A} = \langle A_x, A_y \rangle$ notation for vectors. The component A_x has the interpretation as the amount the vector \vec{A} is in the x -direction and A_y in the y -direction. This angled-bracket notation is nonstandard for physics texts but has become the norm with moderns calculus texts. The traditional approach of physics texts at this level is two write vectors in terms of the basis unit vectors $\hat{x} = \hat{i}$ and $\hat{y} = \hat{j}$. We are using the standard notation for unit vectors putting a " ^ " over the top of the unit vector.



$$\vec{A} = \langle A_x, A_y \rangle = A_x \hat{x} + A_y \hat{y} = A_x \hat{i} + A_y \hat{j} \text{ where } \hat{x} = \hat{i} = \langle 1, 0 \rangle \text{ and } \hat{y} = \hat{j} = \langle 0, 1 \rangle$$

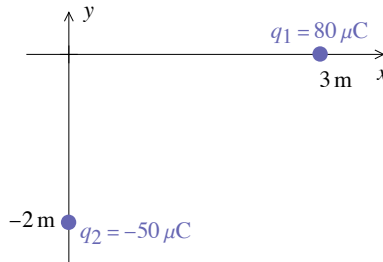
The magnitude of a vector \vec{A} is written as the variable without the arrow A or using the bracket $\|\vec{A}\|$. The magnitude is found applying the Pythagorean theorem to the components.

$$A = \|\vec{A}\| = \sqrt{A_x^2 + A_y^2}$$

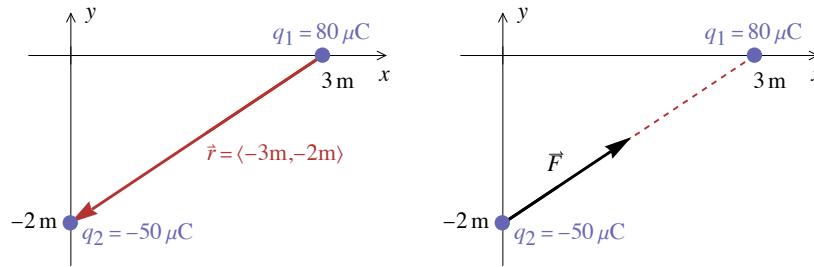
For vectors in three-dimensions, just add a third component $\vec{A} = \langle A_x, A_y, A_z \rangle$ and a third basis unit vector $\hat{z} = \hat{k} = \langle 0, 0, 1 \rangle$; the magnitude becomes $\sqrt{A_x^2 + A_y^2 + A_z^2}$. The best way to specify a direction is to use a unit vector; this works in 2D or 3D.

Example A.2 - Two Point Charges

An $80 \mu\text{C}$ is at $(3 \text{ m}, 0)$ and a $-50 \mu\text{C}$ is at $(0, -2 \text{ m})$. What is the force on the $-50 \mu\text{C}$ charge?



Solution



The vector from $Q_1 = 80 \mu\text{C}$ to $Q_2 = -50 \mu\text{C}$ is $\vec{r} = \langle -3 \text{ m}, -2 \text{ m} \rangle$, where we choose the vector from the other charge to the charge we are calculating the force on. Two equivalent forms of Coulomb's force law are

$$\vec{F} = k_e Q_1 Q_2 \frac{\hat{r}}{r^2} \quad \text{and} \quad \vec{F} = k_e Q_1 Q_2 \frac{\vec{r}}{r^3}.$$

Since these are equivalent one should choose the form that is easiest. If the vector \vec{r} is along an axis then the unit vector form is easier. As an example, if \vec{r} is in the negative y direction then, $\hat{r} = -\hat{y} = \langle 0, -1 \rangle$. If \vec{r} is in some other direction, then $\hat{r} = \vec{r}/r = \vec{r}/\|\vec{r}\|$ is awkward.

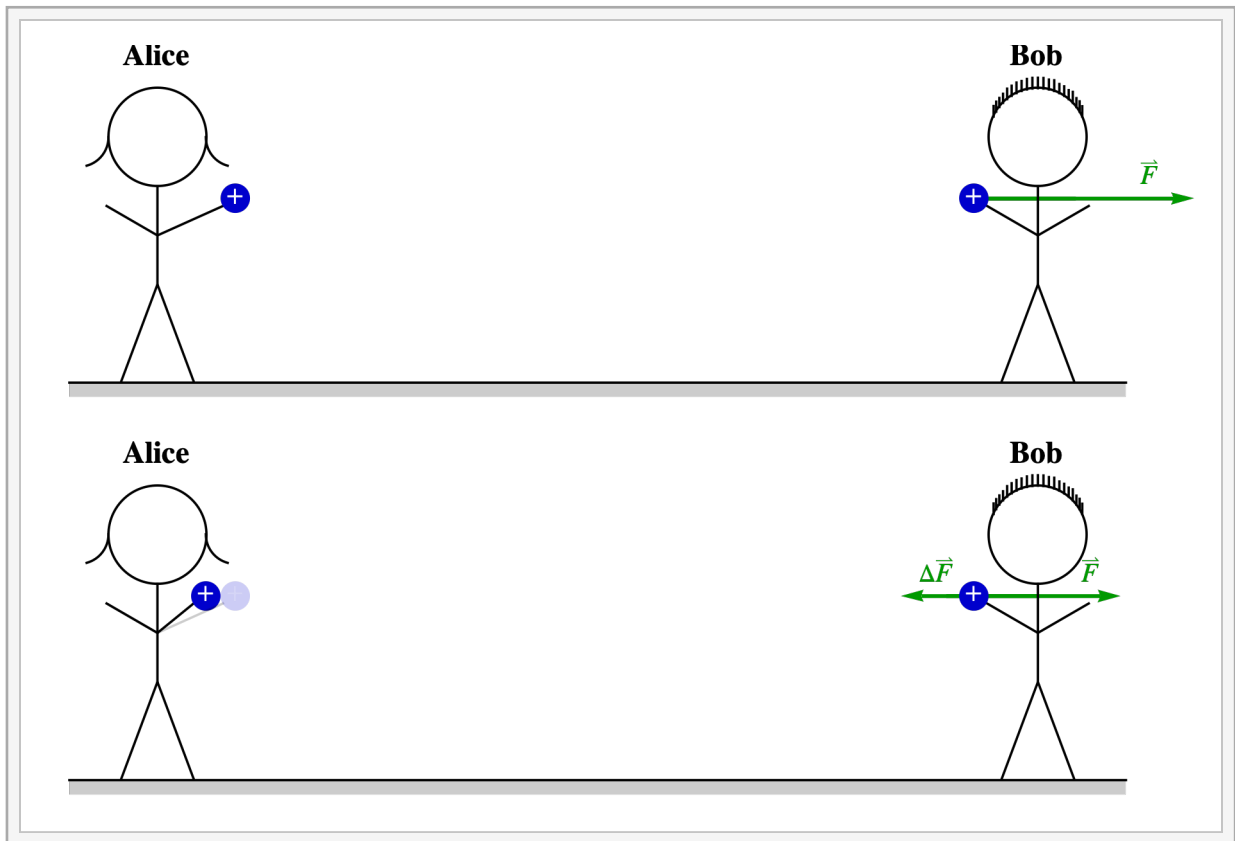
This is the case here, since $\vec{r} = \langle -3 \text{ m}, -2 \text{ m} \rangle$.

$$\begin{aligned} \vec{F} &= k_e Q_1 Q_2 \frac{\vec{r}}{r^3} \\ &= 9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2} (80 \times 10^{-6} \text{ C})(-50 \times 10^{-6} \text{ C}) \frac{\langle -3 \text{ m}, -2 \text{ m} \rangle}{[(3 \text{ m})^2 + (2 \text{ m})^2]^{3/2}} \\ &= \langle 2.30, 1.54 \rangle \text{ N} \end{aligned}$$

Electrostatics and the Need for Fields

Coulomb's law describes the interaction of charged particles. It turns out that this is only correct in the context of electrostatics. Electrostatics is the study of electric charges, electric forces and fields where nothing is allowed to move. Since currents are moving charges, we do not allow for currents in electrostatics. Coulomb's law, and Newton's law of universal gravitation as well, appear to involve forces that acts instantaneously over arbitrarily large distances. This is known as action-at-a-distance. Since the introduction of Newtonian gravity, many philosophers of science were troubled by this notion.

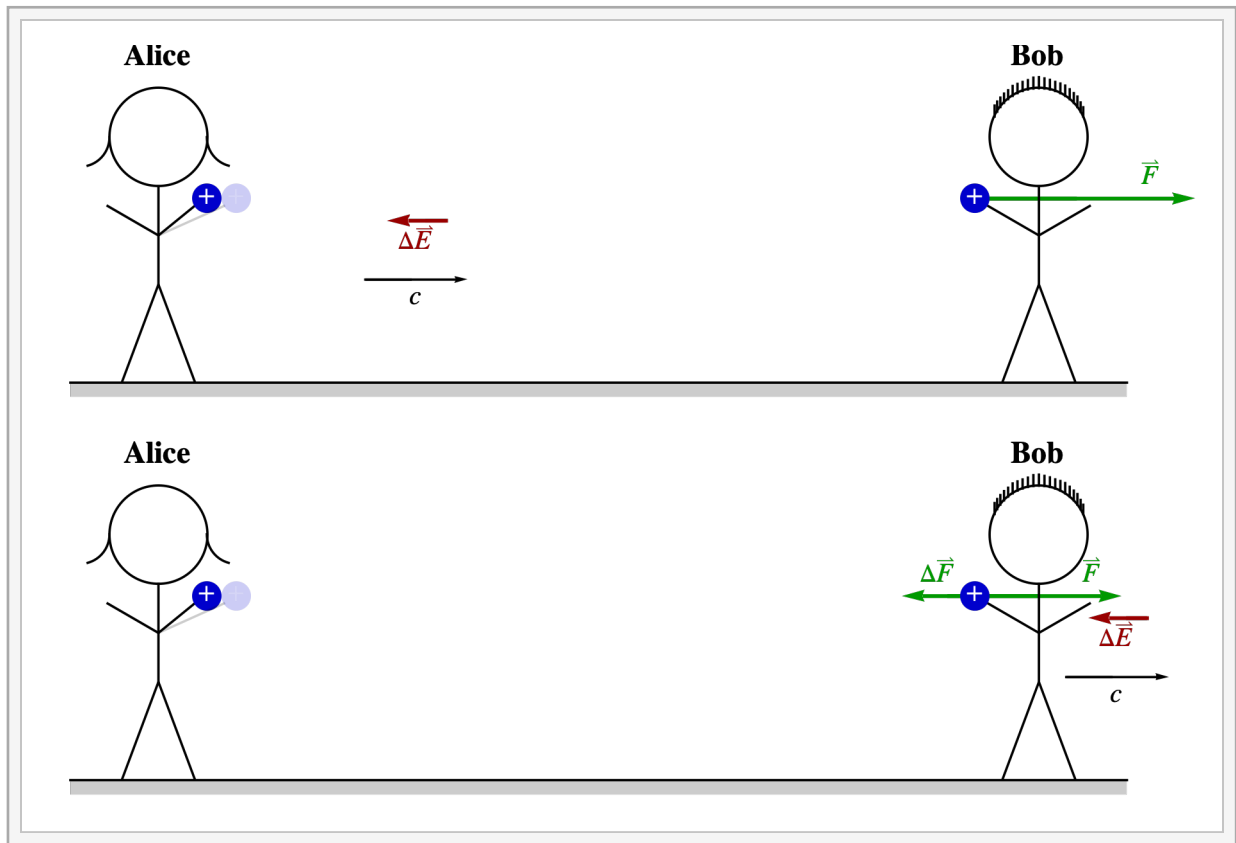
The problem of action-at-a-distance became more serious with Einstein's introduction of the theory of special relativity in 1905. With special relativity, if one can send information faster than light, then one can send information backward in time; this would generate many causal paradoxes, not to mention science fiction plots. By information we mean merely one bit of information, a zero or a one. To demonstrate this in the context of Coulomb's law consider the following. Alice and Bob are some distance apart and both hold electric charges in their hands. If Alice moves her charge away from Bob then, according to Coulomb's law, Bob would instantly feel a decrease in the force on his charge. To see how this can convey information, when Bob feels a smaller force, call this a one and when a larger force call that a zero; thus by moving her charge at regular time intervals, Alice may send information to Bob *instantaneously*.



Interactive Figure - Coulomb's law involves action-at-a-distance. If Alice and Bob both hold charges and Alice moves her charge away from Bob, then Bob would immediately feel the smaller force on his charge.

It turns out however that in the nineteenth century, well before Einstein's introduction of special relativity, the problem of action-at-a-distance was removed for the case of the electric force through the introduction of electric and magnetic fields and through Maxwell's equations. Maxwell's equations describe the behavior of the fields combined with stationary and moving electric charges. With Maxwell's equations, information may not be conveyed faster than light speed.

This is one of the big ideas of physics. Particles do not interact directly; all interaction is mediated by fields. Particles create fields, fields propagate by dynamical rules at a finite speed no faster than the speed of light and then fields exert forces on particles. In the case of electromagnetism, charged particles create electric and magnetic fields, as described by Maxwell's equations. Also, Maxwell's equations show how electric and magnetic field propagate. Electric and magnetic fields then exert forces on charged particles. So for the case of Alice and Bob, when Alice moves her charge, it creates a disturbance in the electromagnetic field (both fields combined). That disturbance propagates at the speed of light and only when that disturbance reaches Bob's position, does he feel a change in the force. It is not instantaneous.



Interactive Figure - Electromagnetism, as described by Maxwell's equations, does not involve action-at-a-distance. When Alice moves her charge, that creates a disturbance in the electromagnetic field, figuratively represented here by $\Delta \vec{E}$, and that disturbance propagates at the speed of light. Bob does not notice the smaller force until that disturbance reaches him.

The issue of action-at-a-distance was still present in Newtonian gravity at Einstein's time; with special relativity, Newton's law of universal gravitation clearly needed to be replaced. Einstein resolved this with a revolutionary new theory, General Relativity, which is a field theory of gravity that describes gravitational attraction in terms of the curvature of space and time.

A.4 - The Electric Field

The gravitational analog of electric field \vec{E} is the gravitational field \vec{g} . To define the gravitational field, we find the force \vec{F} on a test mass m_0 and divide the test mass into it.

$$\vec{g} = \frac{\vec{F}}{m_0}.$$

With a larger test mass the force is larger but when we divide out by the mass the result for the gravitational field is the same.

We define the electric field similarly. Find the force \vec{F} on a test charge q_0 and divide the test charge into it.

$$\vec{E} = \frac{\vec{F}}{q_0}.$$

This is then independent of the value of the the charge. The direction of the electric field is the direction of the force on a positive test charge; thus, it points toward negative charges and away from positive charges.

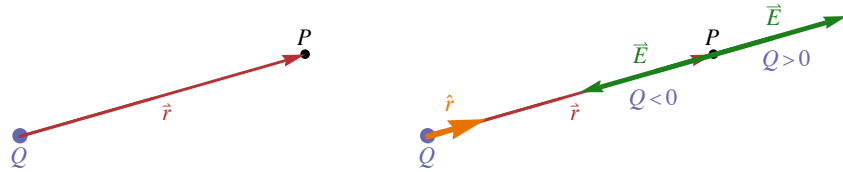
Field of a Point Charge

The electric field for a point charge Q can be found using the prescription above. Take the vector \vec{r} to be the vector from Q to some point P . To get \vec{E} due to Q at P , first find the force on a test charge q_0 placed at P using Coulomb's law.

$$\vec{F} = k_e \frac{Q q_0}{r^2} \hat{r} = k_e \frac{Q q_0}{r^3} \vec{r}$$

Dividing by q_0 gives

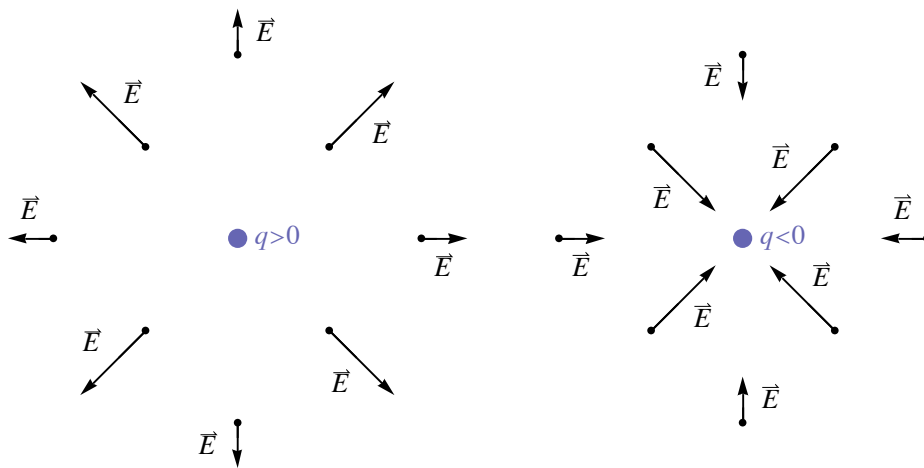
$$\vec{E} = k_e Q \frac{\hat{r}}{r^2} = k_e Q \frac{\vec{r}}{r^3}$$



The magnitude of the field is given by

$$E = \|\vec{E}\| = k_e \frac{|Q|}{r^2}$$

where the absolute value of the charge forces the result to be positive, as the magnitude of a vector must. Since \hat{r} is the radial unit vector (pointing away from the charge) then the direction of \vec{E} is given by $\pm \hat{r}$, where the sign is the same as for Q .



The electric field is directed away from positive charges and toward negative.

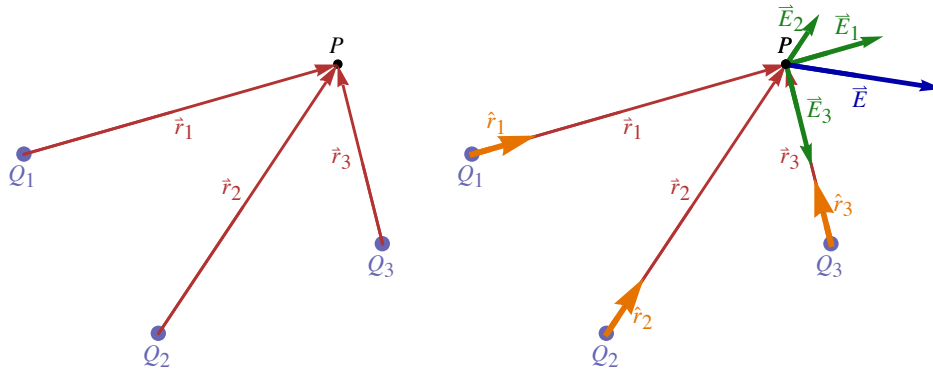
Principle of Superposition

To find the field for more complex charge configurations we use the principle of superposition. A charge distribution may be viewed as a collection of point charges. The electric field due to that distribution is the sum (or integral) over the fields due to point charges. We will consider first the case of a discrete distribution, where there is a collection of point charges, and then the case of a continuous distribution, where the charge is spread continuously over a line, surface or a volume.

Discrete Distribution

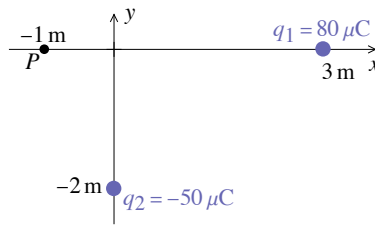
We want the field at a point P due to charges q_i . Take the vector \vec{r}_i to be the vector from q_i to P . \hat{r}_i is the unit vector in the direction of \vec{r}_i . The field \vec{E} is the sum of the fields due to each charge.

$$\vec{E} = k_e \sum_i Q_i \frac{\hat{r}_i}{r_i^2} = k_e \sum_i Q_i \frac{\vec{r}_i}{r_i^3}$$

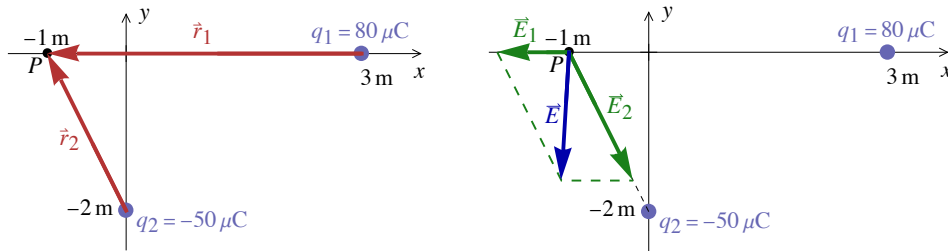


Example A.3 - Two Point Charges (Continued)

An $80\mu\text{C}$ is at $(3\text{ m}, 0)$ and a $-50\mu\text{C}$ is at $(0, -2\text{ m})$. What is the electric field at $(-1\text{ m}, 0)$?



Solution



P is the point where we evaluate the field, so here: $P = (-1\text{ m}, 0)$. The vector from $Q_1 = 80\mu\text{C}$ to P is $\vec{r}_1 = \langle -4\text{ m}, 0 \rangle$ and from $Q_2 = -50\mu\text{C}$ to P we have $\vec{r}_2 = \langle -1\text{ m}, 2\text{ m} \rangle$. where we choose the vector from the other charge to the charge we are calculating the force on. The equivalent forms of the inverse square law for \vec{E}_i are

$$\vec{E}_i = k_e Q_i \frac{\hat{r}_i}{r_i^2} = k_e Q_i \frac{\vec{r}_i}{r_i^3}$$

Of the equivalent forms we choose the easiest. Since the vector \vec{r}_1 is along an axis then the unit vector form is easier for it.

$$\vec{r}_1 = \langle -4\text{ m}, 0 \rangle \Rightarrow \hat{r}_1 = \langle -1, 0 \rangle = -\hat{x} \text{ and } r_1 = 4\text{ m} \Rightarrow \vec{E}_1 = k_e Q_1 \frac{\hat{r}_1}{r_1^2}$$

\vec{r}_2 is not along an axis so the \vec{r}_i/r_i^3 form is easiest.

$$\vec{r}_2 = \langle -1\text{ m}, 2\text{ m} \rangle \Rightarrow \vec{E}_2 = k_e Q_2 \frac{\vec{r}_2}{r_2^3}$$

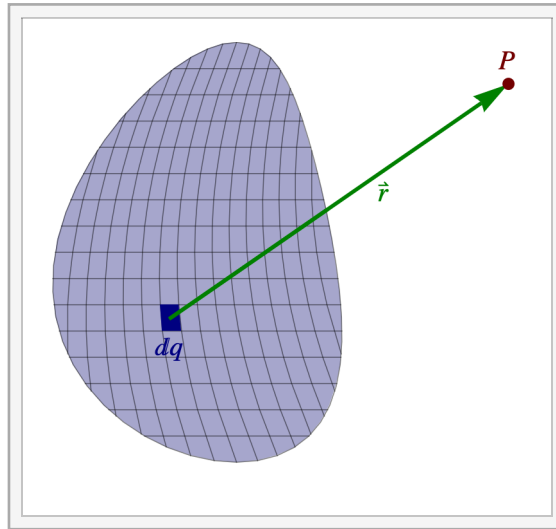
Summing \vec{E}_1 and \vec{E}_2 gives the total electric field.

$$\begin{aligned} \vec{E} &= \vec{E}_1 + \vec{E}_2 = k_e Q_1 \frac{\hat{r}_1}{r_1^2} + k_e Q_2 \frac{\vec{r}_2}{r_2^3} \\ &= 9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2} (80 \times 10^{-6} \text{ C}) \frac{\langle -1, 0 \rangle}{(4\text{ m})^2} \end{aligned}$$

$$\begin{aligned}
& +9.0 \times 10^9 \frac{\text{N}\cdot\text{m}^2}{\text{C}^2} (-50 \times 10^{-6} \text{ C}) \frac{\langle -1 \text{ m}, 2 \text{ m} \rangle}{[(1 \text{ m})^2 + (2 \text{ m})^2]^{3/2}} \\
& = \langle -45\,000, 0 \rangle \text{ N/C} + \langle 40\,250, -80\,500 \rangle \text{ N/C} \\
& = \langle -4750, -80\,500 \rangle \text{ N/C}
\end{aligned}$$

Continuous Distribution

We want the field at a point P due to a continuous distribution of charge. In this case we break up the distribution into an infinite number of infinitesimal pieces, where the charge of an infinitesimal piece is dq .



Summing over the dq consists of evaluating an integral. Take Q to be the total charge; writing this as the sum over all dq gives

$$Q = \int dq.$$

Take the vector \hat{r} to be the vector from dq to P . The field is

$$\vec{E} = k_e \int \frac{\hat{r}}{r^2} dq = k_e \int \frac{\hat{r}}{r^3} dq.$$

The above integrals, although they look like indefinite integrals, are vaguely defined definite integrals, where the integration variables and limits must be determined on a case by case basis.

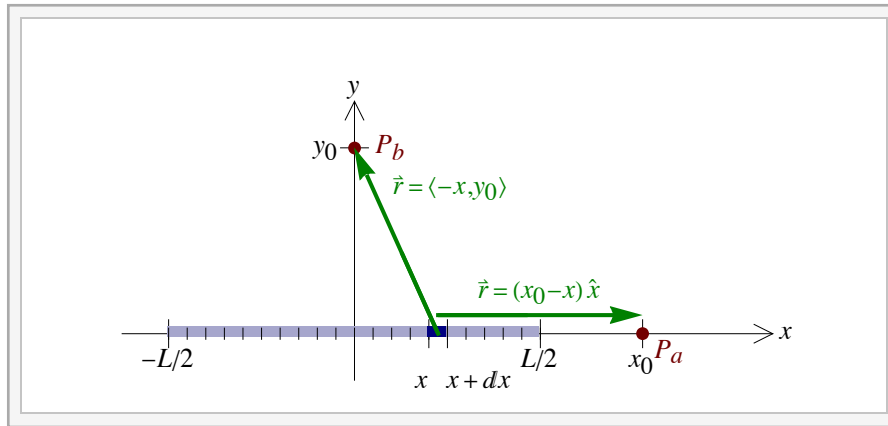
When we specify a continuous charge distribution, we are dealing with insulators. A conductor has freely moving charges; if one puts a charge on a conductor, then that charge will move until electrostatic equilibrium is reached. This will be discussed in Chapter C.

Using Integrals to Calculate Electric Fields

1. Mathematically break up the distribution into an infinite number of infinitesimal pieces. This requires choosing an integration variable and its limits.
2. Find dq , the infinitesimal charge of each infinitesimal piece. You must write the dq in terms of the differential d of the integration variable.
3. Write an expression for \hat{r} , the vector from dq to the point P where the value of the field is wanted. Note that r is the magnitude of \hat{r} .
4. Put the pieces together to form a properly-defined definite integral and evaluate the integral.

Example A.4 - Uniform Line of Charge

Consider a line of uniform charge Q along the x -axis from $-L/2$ to $L/2$.



Interactive Figure

- (a) What is the electric field along the x -axis at x_0 , where $x_0 > L/2$?

Solution

Choose x as the integration variable with limits: $-\frac{L}{2} \leq x \leq \frac{L}{2}$.

The charge between x and $x + dx$ is: $dq = \frac{Q}{L} dx$.

The vector from dq (at x) to x_0 is: $\vec{r} = \langle x_0 - x, 0 \rangle = \hat{x}(x_0 - x)$. Thus: $\hat{r} = \hat{x}$ and $r = \|\vec{r}\| = |x_0 - x| = x_0 - x$

$$\vec{E} = k_e \int \frac{\hat{r}}{r^2} dq = k_e \int_{-L/2}^{L/2} \frac{\hat{x}}{(x_0 - x)^2} \frac{Q}{L} dx = \hat{x} k_e \frac{Q}{L} \int_{-L/2}^{L/2} \frac{dx}{(x_0 - x)^2}$$

This is an integral of the form: $\int u^{-2} du$ where $u = x_0 - x$. Since $du = -dx$ and $\int u^{-2} du = -1/u + C$ we can evaluate the integral. (Note that the signs cancel.)

$$\vec{E} = \hat{x} k_e \frac{Q}{L} \int_{-L/2}^{L/2} \frac{dx}{(x_0 - x)^2} = \hat{x} k_e \frac{Q}{L} \frac{1}{x_0 - x} \Big|_{-L/2}^{L/2} = \hat{x} k_e \frac{Q}{L} \left(\frac{1}{x_0 - L/2} - \frac{1}{x_0 + L/2} \right)$$

- (b) What is the electric field along the positive y -axis at y_0 ?

Solution

It is the same charge distribution so the first two steps are the same: $-\frac{L}{2} \leq x \leq \frac{L}{2}$ and $dq = \frac{Q}{L} dx$.

The vector from dq , at $(x, 0)$ to $(0, y_0)$ is: $\vec{r} = \langle -x, y_0 \rangle = \hat{y} y_0 - \hat{x} x$.

$$\vec{E} = k_e \int \frac{\vec{r}}{r^3} dq = k_e \int_{-L/2}^{L/2} \frac{\langle -x, y_0 \rangle}{(x^2 + y_0^2)^{3/2}} \frac{Q}{L} dx = k_e \frac{Q}{L} \left\langle - \int_{-L/2}^{L/2} \frac{x}{(x^2 + y_0^2)^{3/2}} dx, y_0 \int_{-L/2}^{L/2} \frac{1}{(x^2 + y_0^2)^{3/2}} dx \right\rangle$$

By symmetry the integral for the x -component is zero, leaving only the y -component.

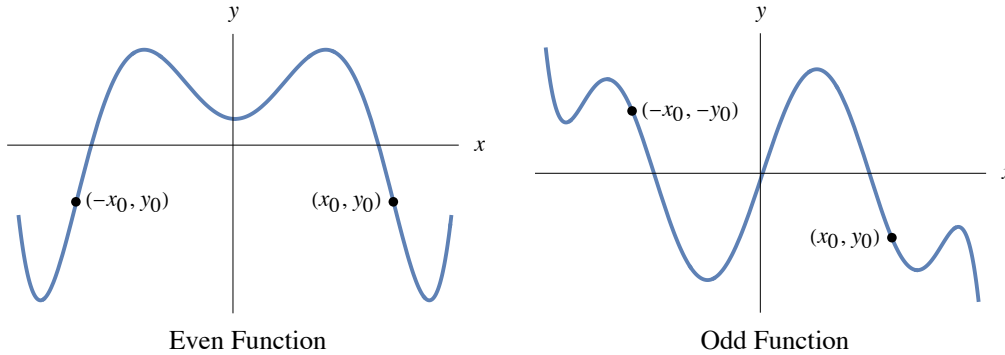
$$\vec{E} = \hat{y} k_e \frac{Q}{L} y_0 \int_{-L/2}^{L/2} \frac{dx}{(x^2 + y_0^2)^{3/2}}$$

This is of the tabulated form: $\int \frac{dx}{(x^2 + a^2)^{3/2}} = \frac{1}{a^2} \frac{x}{\sqrt{x^2 + a^2}} + C$. We can then evaluate the integral.

$$\vec{E} = \hat{y} k_e \frac{Q}{L} \frac{1}{y_0} \frac{x}{\sqrt{x^2 + y_0^2}} \Big|_{-L/2}^{L/2} = \hat{y} k_e \frac{Q}{y_0} \frac{1}{\sqrt{(L/2)^2 + y_0^2}}$$

We used symmetry to argue that the x -component of the field vanishes. We can see mathematically that it vanished because the integral is an odd function integrated between symmetric limits. The integrand $x/(x^2 + y_0^2)^{3/2}$ is odd because it changes sign when the sign of x is changed. This is clarified in the next subsection.

A function f is $\begin{cases} \text{even} \\ \text{odd} \end{cases}$ when $f(-x) = \begin{cases} f(x) \\ -f(x) \end{cases}$. The graph of an even function is symmetric under a reflection through the y -axis and an odd function is symmetric under a 180° rotation about the origin. For an even function, if (x_0, y_0) is on the graph then so is $(-x_0, y_0)$. For an odd function, if (x_0, y_0) is on the graph then $(-x_0, -y_0)$ is as well.



Any function f can be written as the sum of an even and odd functions.

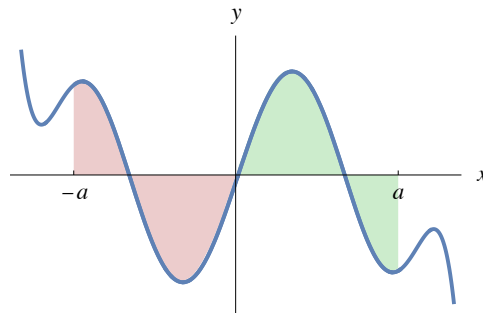
$$f(x) = f_{\text{even}}(x) + f_{\text{odd}}(x) \text{ where } f_{\text{even}}(x) = \frac{1}{2}(f(x) + f(-x)) \text{ and } f_{\text{odd}}(x) = \frac{1}{2}(f(x) - f(-x))$$

As an example: for any polynomial function, its even part is the sum of all even power terms and the odd part is the sum of all odd powers. Similarly, cosh and sinh are the even and odd parts of the exponential function.

The important result for integration is: the integral of any odd function between symmetric limits must vanish. By symmetric limits, I mean the lower limit is the negative of the upper limit.

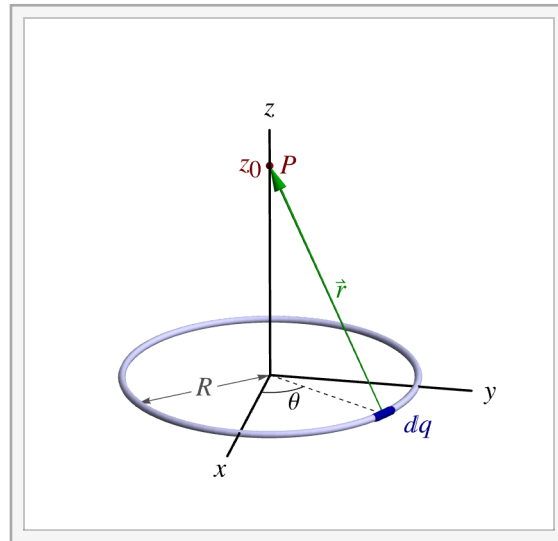
$$\int_{-a}^a f(x) dx = 0 \text{ for any odd } f$$

This can be seen clearly by looking at the graph. (It can be shown rigorously by making the substitution $x \rightarrow -x$.)



The integral of an odd function between symmetric limits vanishes.

Example A.5 - Ring of Uniform Charge



Interactive Figure

Consider a thin ring of radius R with uniform charge Q in the xy -plane with the center at the origin

(a) What is the electric field at the origin?

Solution

Because of symmetry $\vec{E} = \vec{0}$.

(b) What is the electric field along the positive z -axis at z_0 ?

Solution

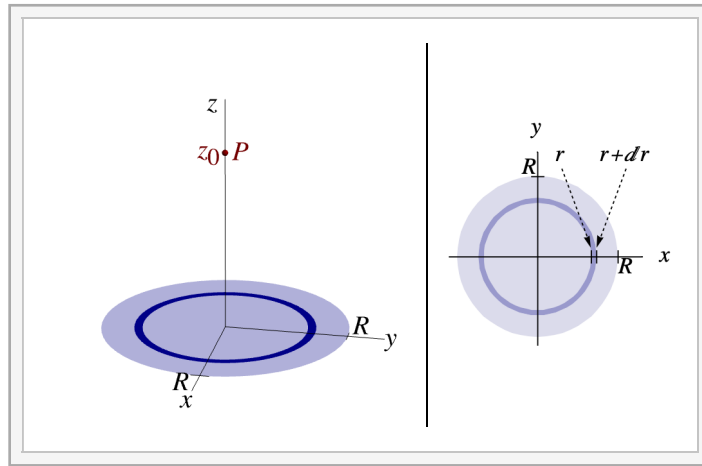
Because of symmetry we can conclude that the field must be in the z -direction.

$$\begin{aligned} \vec{E} &= k_e \int \frac{\vec{r}}{r^3} dq = \hat{z} E_z \\ &= \hat{z} k_e \int \frac{z_0}{(R^2 + z_0^2)^{3/2}} dq = \hat{z} k_e \frac{z_0}{(R^2 + z_0^2)^{3/2}} \int dq = \hat{z} k_e \frac{z_0 Q}{(R^2 + z_0^2)^{3/2}} \end{aligned}$$

Example A.6 - Disk of Uniform Charge

Consider a thin disk of radius R with uniform charge Q in the xy -plane with the center at the origin. What is the electric field along the positive z -axis at z_0 ?

Solution



Interactive Figure

We begin with the field due to a uniform ring.

$$\vec{E} = \hat{z} k_e \frac{z_0 Q}{(R^2 + z_0^2)^{3/2}}$$

We then break up the disk into concentric rings and sum over them by integrating. We will choose our integration variable to be r , varying from 0 to R : $0 \leq r \leq R$. The distance from the center of each ring is the same z_0 and the field is still in the \hat{z} -direction. Each ring now has a radius r and a charge dq . We now need to make the following substitutions:

$$Q \rightarrow dq, \quad R \rightarrow r, \quad z_0 \rightarrow z_0 \quad \text{and} \quad \hat{z} \rightarrow \hat{z}$$

and we get

$$\vec{E} = \hat{z} k_e \int \frac{z_0 dq}{(r^2 + z_0^2)^{3/2}}$$

The charge between r and $r + dr$ is dq . With a uniform charge, the fraction of the charge is the same as the fraction of the area $dq/Q = dA/A_{\text{tot}}$. The (infinitesimal) area is the length of the curve, the circumference $2\pi r$ times dr , the thickness: $dA = 2\pi r dr$. The total area is $A_{\text{tot}} = \pi R^2$. This gives us dq .

$$dq = \frac{Q}{A_{\text{tot}}} dA = \frac{Q}{\pi R^2} 2\pi r dr = \frac{2Q}{R^2} r dr$$

$$\vec{E} = \hat{z} k_e \int \frac{z_0 dq}{(r^2 + z_0^2)^{3/2}} = \hat{z} k_e z_0 \frac{2Q}{R^2} \int_0^R \frac{r dr}{(r^2 + z_0^2)^{3/2}}$$

This is an integral of the form: $\int u^{-3/2} du$

$$u = r^2 + z_0^2 \implies du = 2r dr \quad \text{and} \quad \int u^{-3/2} du = \frac{1}{-1/2} u^{-1/2} + C = -2u^{-1/2} + C$$

We can then evaluate the integral

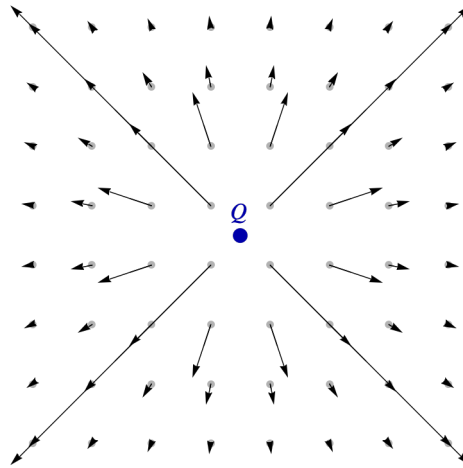
$$\vec{E} = -\hat{z} k_e z_0 \frac{2Q}{R^2} \frac{1}{\sqrt{r^2 + z_0^2}} \Bigg|_0^R = -\hat{z} k_e z_0 \frac{2Q}{R^2} \left(\frac{1}{\sqrt{R^2 + z_0^2}} - \frac{1}{z_0} \right)$$

and then simplify it.

$$\vec{E} = \hat{z} k_e \frac{2Q}{R^2} \left(1 - \frac{z_0}{\sqrt{R^2 + z_0^2}} \right)$$

Field Diagrams

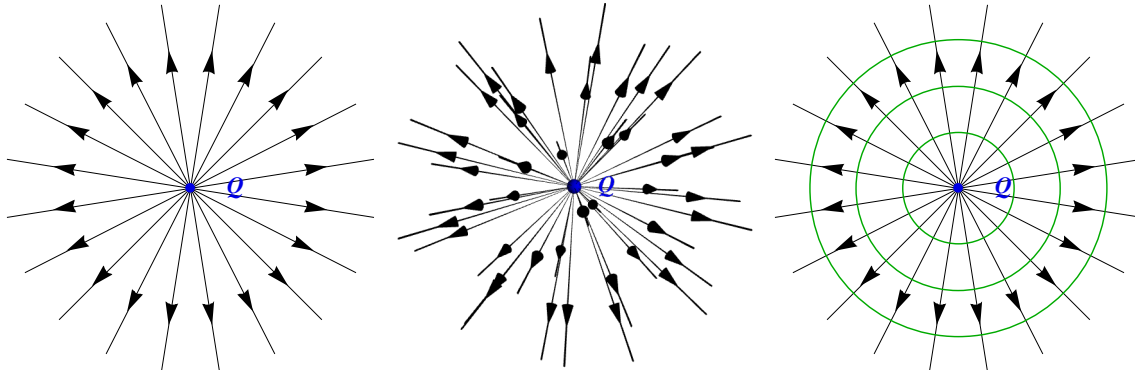
The electric field is a vector field. This means that at each position in space there is a vector. The usual way of representing a general vector field is to draw a grid with a vector at each point in the grid.



Point Charge Field - Vectors on a Grid

Such diagrams, however, can get quite complicated in the case of electric fields; one gets a tangled mess of overlapping arrows. The convention that is used is to draw continuous curves showing only the direction of the field at some position. The field lines begin at positive charges and end at negative charges. We will see that the density of lines is a measure of the strength (magnitude) of the field.

Consider the field of a positive point charge; the field lines point radially away from the charge.



Left: Point Charge Field in 2D - continuous curves Center: 3D version of same Right: Concentric sphere around the charge

Different concentric spheres with the charge at the center will have the same number of lines passing through them. The area of a sphere is $A = 4\pi r^2$, so if we take the number of lines per area we get:

$$\frac{\text{\# of lines}}{\text{Area}} \propto \frac{1}{r^2}$$

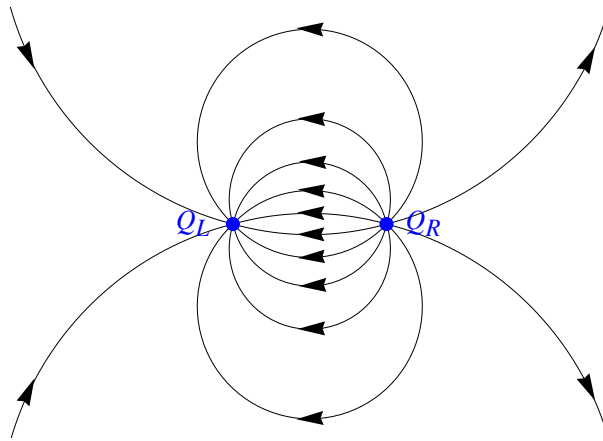
where " \propto " is the proportional symbol. The electric field magnitude is also proportional to $\frac{1}{r^2}$. It follows that

$$\text{mag of field} = E \propto \frac{\text{\# of lines}}{\text{Area}}$$

This gives the graphical interpretation of the magnitude of the field. When the lines are close together the field is strong and when they are far apart it is weak.

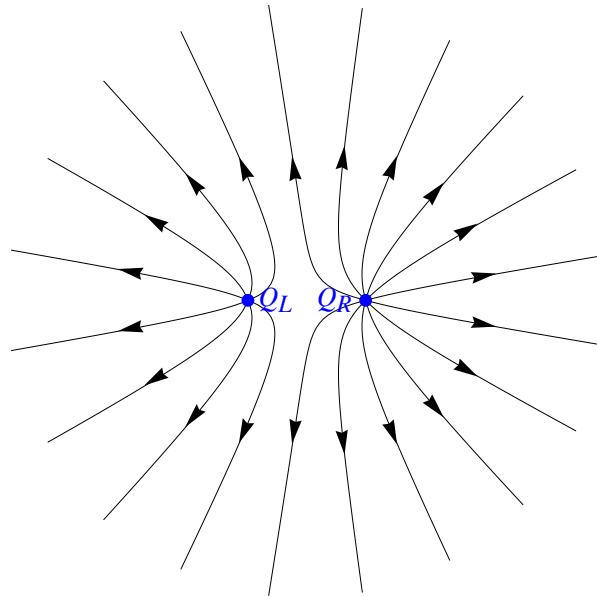
Electric Dipole

$Q_L = -Q$ is the charge on the left and $Q_R = +Q$ is the charge on the right.



Two Positive Charges

$Q_L > 0$ is the charge on the left and $Q_R > 0$ is the charge on the right. Also $Q_L < Q_R$.



One Positive, One Negative, Net Charge Negative

$Q_L < 0$ is the charge on the left and $Q_R > 0$ is the charge on the right. Also $Q_L + Q_R < 0$.

