Chapter B

Gauss's Law

Blinn College - Physics 2326 - Terry Honan

B.1 - Electric Flux

Electric flux is a measure of the number of field lines passing through a surface. We will develop a definition of flux gradually, starting with special cases and generalizing.

Flux of a Uniform Field through a Flat Surface

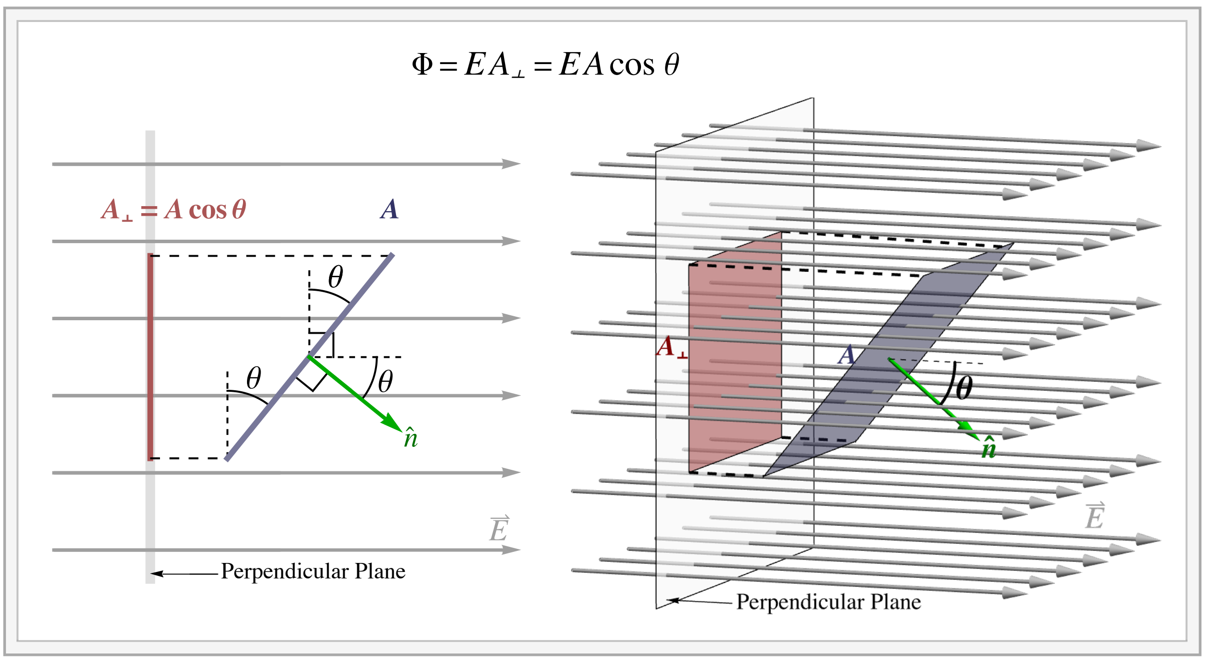
We saw in the discussion of electric field diagrams that the number of field lines per area is a measure of the strength of the field. Strictly, by area we mean $A_\perp$, the part of the area perpendicular to the field.

$$E \propto \frac{\text{\# of lines}}{A_\perp}.$$

We want the flux Φ to be a measure of the number of lines through a surface $\Phi \propto (\text{\# of lines})$ so we can define the flux in the case of a uniform field and a flat surface to be

$$\Phi = E\,A_\perp = E\,A\cos\theta.$$

There is not a unique angle between a vector and a surface, so the angle we use is the angle between the normal to the surface and the field. θ is the angle between the electric field and the normal (perpendicular) to the surface.



Recall the dot product. For any two vectors $\vec{A}$ and $\vec{B}$, their dot product is given by

$$\vec{A}\cdot\vec{B} = A\,B\cos\theta = A_x\,B_x + A_y\,B_y + A_z\,B_z.$$

The expression for flux can then be written in terms of the dot product

$$\Phi = \vec{E}\cdot\vec{A},$$

where the vector $\vec{A}$ is defined to have magnitude A, the area and to be in the direction $\hat{n}$, the unit vector normal to the surface.

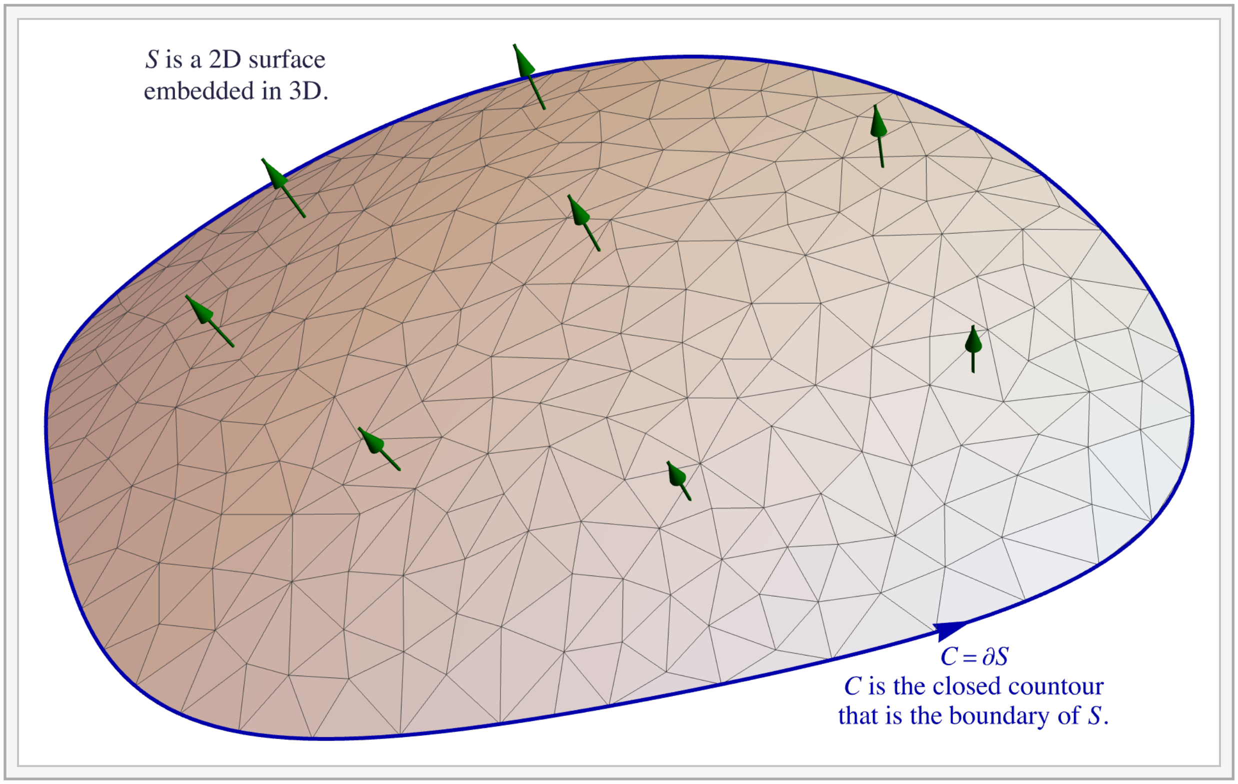
$$\vec{A} = A\,\hat{n}$$

Note that there are two unit normals to any flat surface; if $\hat{n}$ is a unit normal, then so is $-\hat{n}$. This gives a sign ambiguity in the flux. This, we will see, is a general feature, except in the case of a closed surface where we can choose the outward normal.

Flux in General

The expression for flux must now be generalized to the case of a general field that varies spatially and a surface that is not flat. To do this, break the surface into many small (infinitesimal) flat pieces with area vectors given by $d\vec{A} = \hat{n}\, dA$, where dA is the infinitesimal area of the surface and $\hat{n}$ is the unit normal at that position. For sufficiently small pieces the field will be uniform to a good approximation. For one such flat piece, the infinitesimal flux is $\vec{E} \cdot d\vec{A}$. Summing over all the small flat pieces gives an integral. The general definition of flux is then:

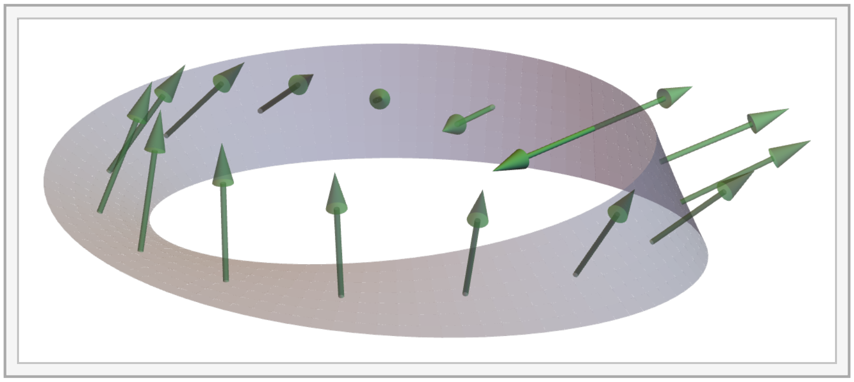
$$\Phi = \int \vec{E} \cdot d\vec{A}.$$



Approximate any surface by dividing it into many small polygons; here we are using triangles.

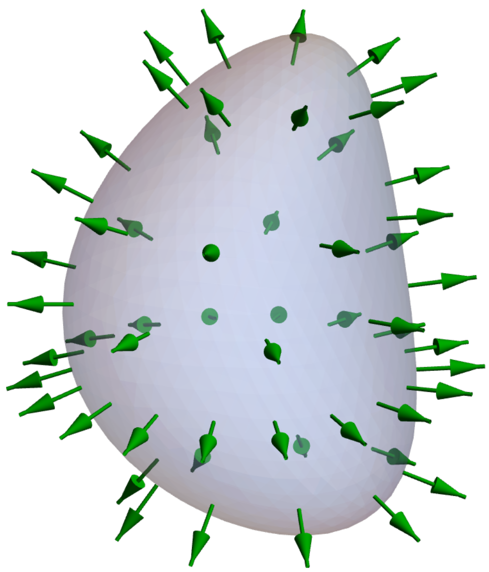
As one moves along a surface S the direction of $\hat{n}$ varies. At every position there are two possible unit normals, differing by a sign. One can continuously move a unit normal around some surface. If doing this traces out a consistent family of normals then that family of normals is called an orientation and the surface is said to be orientable. For an orientable surface there are two possible choices of orientation. C, the boundary of a closed surface, is the curve around its edge. The orientation of a curve is the sense of integration around the curve. The orientation of C is related to the orientation of S by the right-hand-rule; put your thumb in the direction of the unit normals that give the orientation of S and the fingers wrap in the direction of the orientation of C.

As an esoteric mathematical point, it should be mentioned that not all surfaces allow a choice of orientation. These surfaces are called nonorientable; a Möbius strip is an example. Nonorientable surfaces are not important for our purposes and we can always define the flux through a nonorientable surface to be zero.



A Möbius strip is an example of a non-orientable surface.

A closed surface has no boundary. We can remove the sign ambiguity in orientation for closed surfaces by choosing the outward-pointing family of normals.



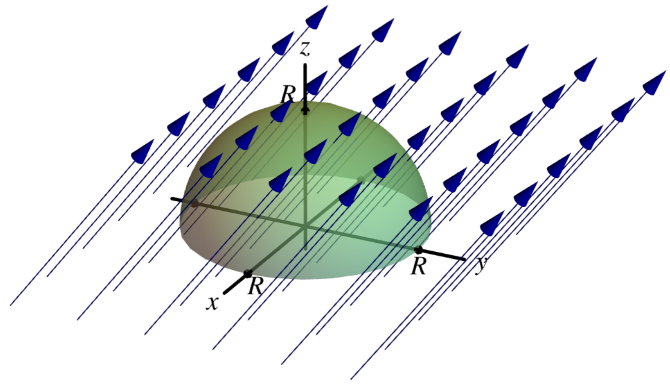
A closed surface has no boundary. We choose the outward family of normals to define its orientation.

Example B.1 - Uniform Field and a Hemisphere

Consider a uniform electric field $\vec{E} = \langle a, b, c \rangle$ and the hemisphere,

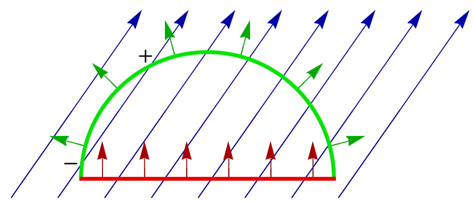
$$x^2 + y^2 + z^2 = R^2 \text{ and } z \geq 0 ,$$

which describes to a hemisphere of radius R above the xy-plane, where the positive-z direction is upward. What is the electric flux through the hemisphere?



Solution

Remember that flux is a measure of the number of field lines passing through a surface. Any field line that passes through the base also passes through the hemisphere. There are some field lines that pass through the hemisphere and not the base but they must both enter and leave the surface, and thus contribute zero to the flux.



It follows that the flux through the hemisphere is the same as the flux through the disk at the base of the hemisphere and the disk is a flat surface. Since the disk is in the xy-plane, the unit normal is $\hat{z}$. The area of the disk is πR^2.

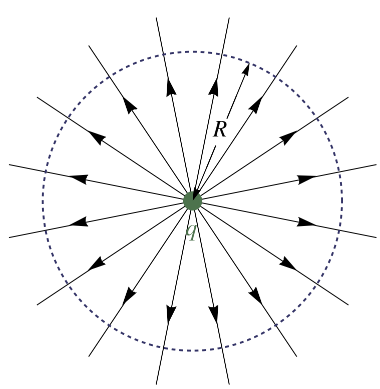
$$\Phi_{\text{hemisphere}} = \Phi_{\text{disk}} = \vec{E} \cdot \vec{A}_{\text{disk}} = \vec{E} \cdot \hat{z}\, \pi R^2 = E_z\, \pi R^2 = c\, \pi R^2$$

Here we have chosen the outward normal for the hemisphere and that corresponds to the upward unit normal for the disk. If we chose the inward normal then the disk's unit normal is downward and our answer would have the opposite sign.

B.2 - Gauss's Law

Gauss's law relates the flux through a closed surface, called a Gaussian surface, to the total charge enclosed by the surface; they are proportional. For a closed surface we can eliminate the sign ambiguity in the flux by choosing the orientation associated with the outward normals.

Point Charge at the Center of a Sphere



We will develop Gauss's law gradually, first considering the case of a point charge q at the center of a spherical Gaussian surface of radius R.

$$\Phi = \oint \vec{E} \cdot d\vec{A} = k_e\, q \oint \frac{\hat{r}}{r^2} \cdot \hat{n}\, dA$$

A circle in the integral sign indicates that the integral is over a closed surface. Here, the integral is over a sphere of radius R. $\hat{n}$ is the unit normal to the sphere, but that is the same as the unit radial vector $\hat{r}$.

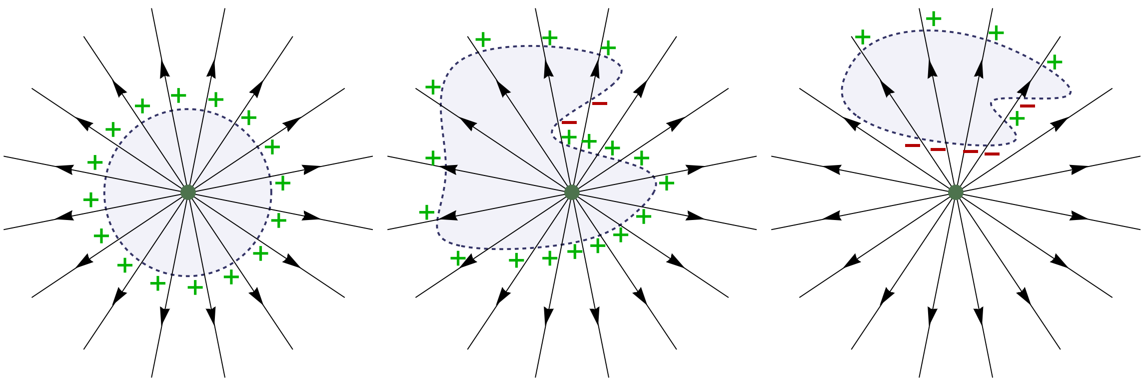
$$r = R = \text{constant} \quad \text{and} \quad \hat{r} = \hat{n} \implies \hat{r} \cdot \hat{n} = \hat{r} \cdot \hat{r} = 1$$

$$\Phi = k_e\, q \oint \frac{\hat{r}}{r^2} \cdot \hat{n}\, dA = \frac{k_e\, q}{R^2} \oint dA = \frac{k_e\, q}{R^2} \times 4\pi R^2$$

$$= 4\pi k_e\, q = \frac{q}{\varepsilon_0}$$

Note that this is independent of the radius of the sphere; this must be the case because any field line that passes through one sphere will pass through another.

Point Charge with Any Closed Surface



Considering any closed surface containing the charge, we can extend the previous result. Recall that flux is a measure of the number of field lines passing through a surface. A field line that passes through a sphere will also pass through any closed surface containing the charge. Note that when a field line enters a surface there is a negative contribution to the flux and a positive contribution when it leaves. A line that both enters and leaves has no net effect on the flux. It follows that any closed surface that contains the charge gets a flux Q/ε_0. Moreover any closed surface that does not contain the charge get a flux of zero.

Gauss's Law

If we now consider all the charges in the universe: $Q_1,\ Q_2,\ Q_3, \ldots$ and their electric fields $\vec{E}_1,\ \vec{E}_2,\ \vec{E}_3, \ldots$. The total electric field is then a sum

$$\vec{E} = \sum_i \vec{E}_i$$

and since the integral of a sum is the sum of the integrals we can write the total flux as a sum

$$\oint \vec{E} \cdot d\vec{A} = \sum_i \oint \vec{E}_i \cdot d\vec{A}$$

If Q_i is inside the surface then $\oint \vec{E}_i \cdot d\vec{A} = Q_i/\varepsilon_0$; if it is outside the integral gives zero. It follows that if Q_{enclosed} is the sum of all charge inside the surface, then we have Gauss's law.

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0}.$$

Mathematically, Gauss's Law is equivalent to Coulomb's law in the case of electrostatics, where no charges are allowed to move. Here, Coulomb's law refers generally to the inverse square law for electric fields and forces. The equivalence means that one implies the other. We have just seen, at least loosely, that Coulomb's law implies Gauss's law. The reverse will be shown below, using spherical symmetry. Despite this mathematical equivalence, at the level of this course we will only be able to find electric fields from Gauss's law in cases of symmetry.

B.3 - Finding Electric Fields in Cases with Symmetry

The general steps for using symmetry to find the field at some point P in cases of symmetry follow. They are somewhat vague, but will be clarified by further examples.

Identify the symmetry and its effect on $\vec{E}$.

A symmetry in the charge distribution affects the possible form of the electric field. The three types of symmetry we will discuss are: spherical, cylindrical and planar.

Choose a Gaussian surface that reflects the symmetry and passes through the point P. $\oint \vec{E} \cdot d\vec{A} = E \times (\text{some area})$

If we have spherical, cylindrical or planar symmetry then a spherical, cylindrical or planar surface will be perpendicular to the field and the magnitude of the field will be uniform over the surface. The flux through that surface will then be $E\,A$. In the cylindrical and planar cases these surfaces will not be closed surfaces, so they cannot be Gaussian surfaces. We can make these surfaces closed by adding pieces that are parallel to the field and then have no contribution to the flux.

Find the charge enclosed by the Gaussian surface.

Q_{enclosed} is the total charge enclosed by the Gaussian surface. Often one must consider multiple cases.

Gauss's Law then gives $\vec{E}$.

The expressions found by the procedure sketched above can be plugged into Gauss's law to give the desired expression for the electric field.

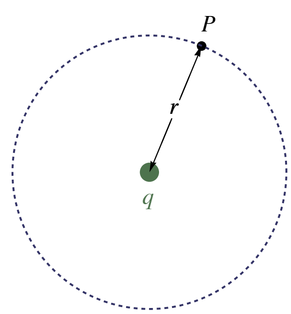
Charge Densities

Often instead of specifying a given charge we talk about charge densities. There are three different charge densities to consider.

- $\lambda = \frac{\text{charge}}{\text{length}}$ = linear charge density $\Longleftrightarrow$ charge $= \lambda \times (\text{some length})$
- $\sigma = \frac{\text{charge}}{\text{area}}$ = surface charge density $\Longleftrightarrow$ charge $= \sigma \times (\text{some area})$
- $\rho = \frac{\text{charge}}{\text{volume}}$ = volume charge density $\Longleftrightarrow$ charge $= \rho \times (\text{some volume})$

Spherical Symmetry

Field of a Point Charge



The charge distribution of a point charge is spherically symmetric. If the charge position is taken as the origin then the charge distribution is the same under any rotation. This places important restrictions on the form of the field. The field at some point P must be in the radial direction; this follows from the fact that the field must be unchanged under some rotation about the line connecting the charge at the origin and P. Also, the field's magnitude must only vary with r, the distance from the origin to P. The general form of the field is:

$$\vec{E} = E_r(r)\,\hat{r} = E_r\,\hat{r}$$

where E_r is the radial component of the field with depends only on the radial distance from the charge r. We need to evaluate the integral over a spherical Gaussian surface passing through P, which is a sphere of radius r. The unit normal vector is the same as the unit radial vector, $\hat{r} = \hat{n}$, which implies that $\hat{r} \cdot \hat{n} = \hat{r} \cdot \hat{r} = 1$. $E_r(r)$ is constant on the sphere and $\oint dA$ is just the surface area of the sphere.

$$\oint \vec{E} \cdot d\vec{A} = \oint E_r\,\hat{r} \cdot \hat{n}\,dA = E_r \oint dA = E_r\,4\,\pi\,r^2$$

The only charge inside the Gaussian surface is the point charge, $Q_{\text{enclosed}} = q$, so

$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E_r\, 4\,\pi\, r^2 = \frac{q}{\varepsilon_0} \implies E_r = \frac{1}{4\,\pi\,\varepsilon_0}\frac{q}{r^2} \implies \vec{E} = E_r\,\hat{r} = \frac{1}{4\,\pi\,\varepsilon_0}\frac{q}{r^2}\,\hat{r} = k_e\,\frac{q}{r^2}\,\hat{r}.$$

General Spherical Symmetry

The general problem of spherical symmetry follows by similar reasoning. The general form of the field is, by symmetry, the same as for the point charge $\vec{E} = E_r(r)\,\hat{r} = E_r\,\hat{r}$. The flux also follows similarly $\oint \vec{E} \cdot d\vec{A} = E(r)\,4\,\pi\,r^2$ and the electric field becomes

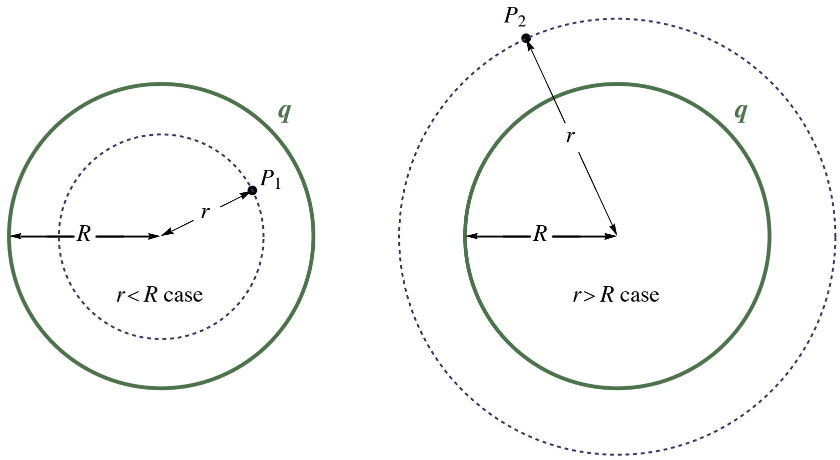
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E_r\, 4\,\pi\, r^2 = \frac{1}{\varepsilon_0}\,Q_{\text{enclosed}} \implies \vec{E} = E_r\,\hat{r} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r}.$$

The value of Q_{enclosed} varies on a case-by-case basis.

Example B.2 - Uniform (Insulating) Spherical Shell

What is the electric field as a function of position for a spherical shell of radius R with uniform charge q? Give answers for both cases: $r < R$ and $r > R$.

Solution

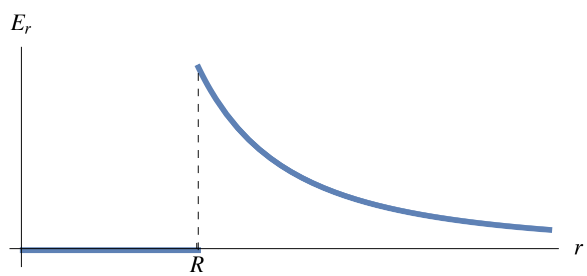


Consider the points P_1 inside the shell ($r < R$) and P_2 outside the shell ($r > R$). For the Gaussian surface through P_1 there is no charge enclosed by the Gaussian surface so $Q_{\text{enclosed}} = 0$. For the Gaussian surface through P_2 the shell's charge q is enclosed by the Gaussian surface so $Q_{\text{enclosed}} = q$

$$r < R: \quad Q_{\text{enclosed}} = 0 \implies \vec{E} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r} = \vec{0}$$

$$r > R: \quad Q_{\text{enclosed}} = q \implies \vec{E} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r} = k_e\,\frac{q}{r^2}\,\hat{r}.$$

Because Q_{enclosed} has a discontinuity at $r = R$, the radial component of the field E_r also has a discontinuity at R.

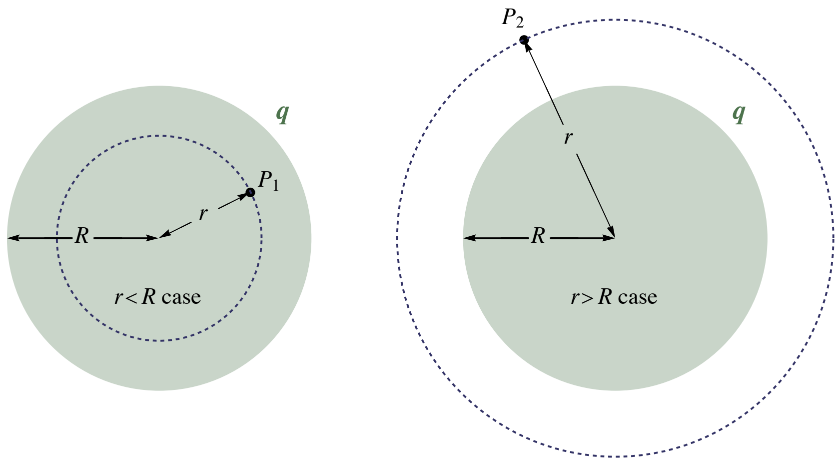


If alternatively you were given the surface charge density σ instead of the charge q, then you would replace q with $4\,\pi\,R^2\,\sigma$.

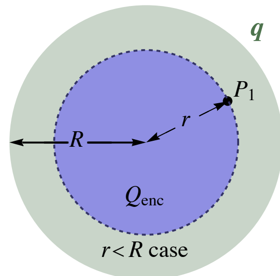
Example B.3 - Uniform (Insulating) Solid Sphere

What is the electric field as a function of position for a solid sphere of radius R with uniform charge q? Give answers for both cases: $r < R$ and $r > R$.

Solution



Consider again the points P_1 inside the shell ($r < R$) and P_2 outside the shell ($r > R$). For the Gaussian surface through P_2 the total charge q is enclosed by the Gaussian surface so $Q_{\text{enclosed}} = q$. For the Gaussian surface through P_1 the Q_{enclosed} is more involved; the fraction of the volume is the fraction of the charge. The volume enclosed V_{enclosed} the Gaussian surface is the volume of a sphere of radius r and the total volume is the volume of a sphere of radius R.

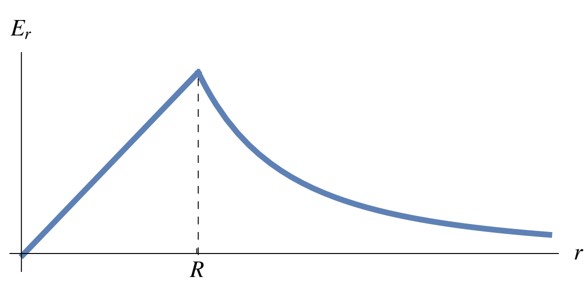


For $r < R$, Q_{enclosed} is the charge inside a sphere of radius r.

$$r < R: \quad V_{\text{enclosed}} = \frac{4}{3}\pi r^3 \text{ and } V_{\text{total}} = \frac{4}{3}\pi R^3 \implies Q_{\text{enclosed}} = q\,\frac{V_{\text{enclosed}}}{V_{\text{total}}} = q\,\frac{r^3}{R^3} \implies \vec{E} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r} = k_e\,\frac{q}{R^3}\,r\,\hat{r}$$

$$r > R: \quad Q_{\text{enclosed}} = q \implies \vec{E} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r} = k_e\,\frac{q}{r^2}\,\hat{r}.$$

Notice that the field is continuous at R.



If alternatively you were given the volume charge density ρ instead of the charge q, then you would replace q with $\frac{4}{3}\pi R^3 \rho$ in the expressions above.

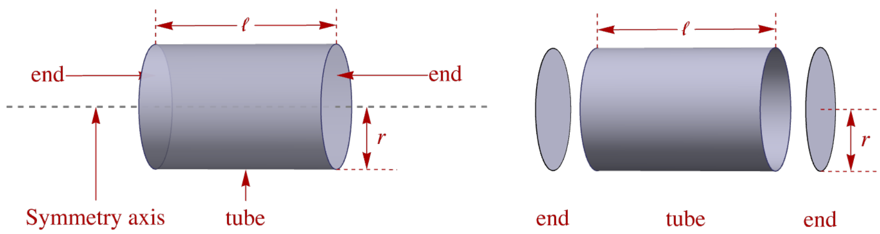
Cylindrical Symmetry

Cylindrical symmetry is the case where there is a rotational symmetry about an axis and a translational symmetry along the same axis; this means if you rotate about an axis everything looks the same and as you move along the same axis nothing changes. Cylindrical symmetry must be infinite. To specify the charge distribution we must use a charge density instead of giving a charge, because a finite charge spread over an infinite volume gives zero charge in any finite region. The field must point radially away from the symmetry axis. The field has the general form

$$\vec{E} = E_r(r)\,\hat{r} = E_r\,\hat{r}$$

which looks the same as the spherical result but is different. Here $\hat{r}$ is the unit vector away from the symmetry axis and r is the perpendicular distance from it. E_r is again the radial component of the field. The cylindrical surface at radius r, which I will refer to as a tube, will then be perpendicular to the field. This tube cannot be a Gaussian surface in itself because it isn't closed. We can make it a closed surface by adding flat ends to the tube. The field is parallel to the ends and the flux through them is zero. Take the length of the tube to be ℓ. The charge distribution must be infinite but the Gaussian surface is finite; its length ℓ cannot appear in the answer and must cancel.

$$\oint \vec{E} \cdot d\vec{A} = \int_{\text{tube}} \vec{E} \cdot d\vec{A} + \int_{\text{end 1}} \vec{E} \cdot d\vec{A} + \int_{\text{end 2}} \vec{E} \cdot d\vec{A}$$
$$= E_r(r)\,A_{\text{tube}} + 0 + 0 = E_r(r)\,2\,\pi\,r\,\ell$$

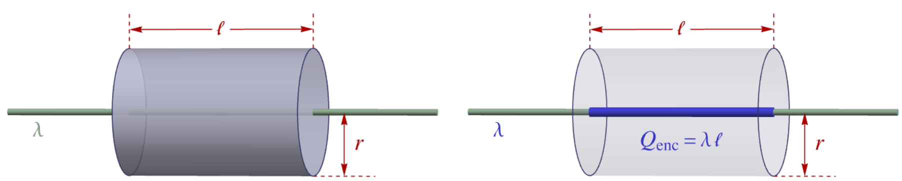


Gaussian Surface for Cylindrical Symmetry - a tube with two ends

Example B.4 - Infinite (Insulating) Line of Charge

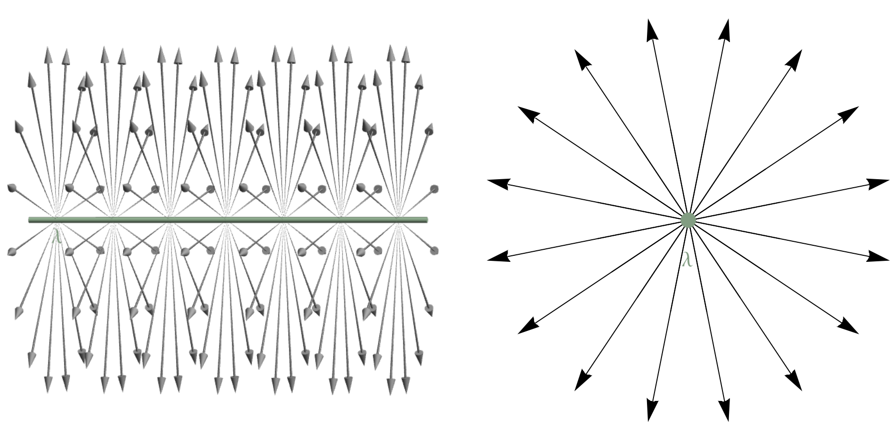
An infinite uniform line of charge has a linear charge density of λ. What is the electric field a distance r from the line of charge?

Solution



The line of charge is the axis of symmetry. The ends of the Gaussian surface chop off the Q_{enclosed}. Since λ is the charge per length and there is a length of ℓ inside the Gaussian surface, we get $Q_{\text{enclosed}} = \lambda\,\ell$.

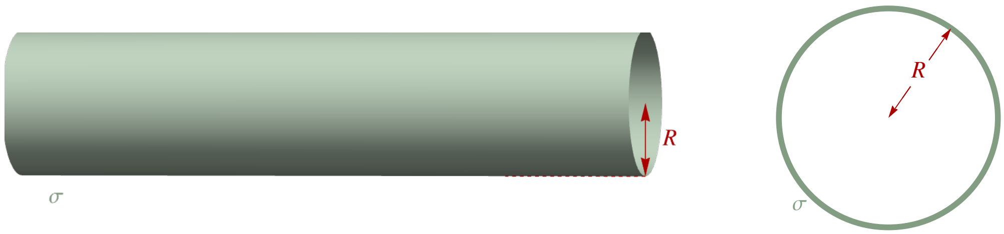
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E_r\,2\,\pi\,r\,\ell = \frac{\lambda\,\ell}{\varepsilon_0} \implies \vec{E} = E_r\,\hat{r} = \frac{1}{2\,\pi\,\varepsilon_0}\,\frac{\lambda}{r}\,\hat{r}.$$



Field lines for infinite line of charge (left) and its cross-section (right)

Example B.5 - Infinite Thin-shelled (Insulating) Cylinder of Charge

An infinite thin-shelled cylinder (tube) of radius R has a uniform surface charge density σ. What is the electric field a distance r from the line of charge? Give answers for both cases: $r < R$ and $r > R$.

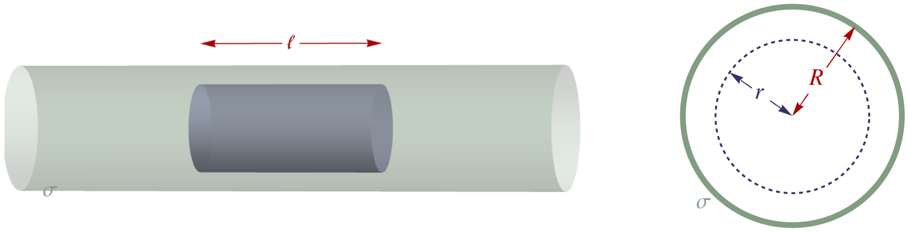


Infinite thin-shelled cylinder (left) and its cross-section (right)

Solution

The central axis of the long cylinder is the axis of symmetry.

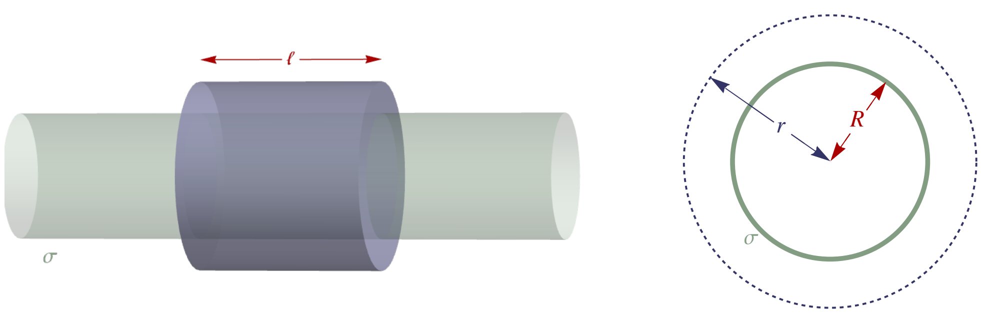
***r* < *R* case:** This the trivial case. The Gaussian surface is entirely inside the charge distribution, so $Q_{\text{enclosed}} = 0$.



Infinite thin-shelled cylinder with Gaussian surface for $r < R$ (left) and its cross-section (right)

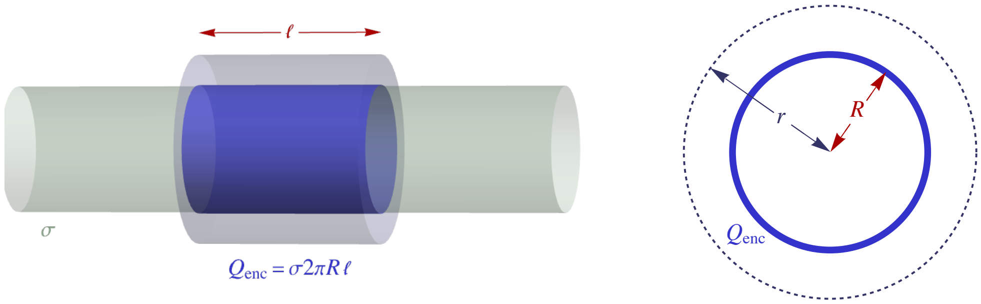
$$\oint \vec{E} \cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E_r\, 2\,\pi\, r\, \ell = \frac{0}{\varepsilon_0} \implies \vec{E} = E_r\,\hat{r} = \vec{0}$$

***r* > *R* case:** Now the Gaussian surface passes outside the charge distribution



Infinite thin-shelled cylinder with Gaussian surface for $r > R$ (left) and its cross-section (right)

To find the Q_{enclosed} we need to identify what of the charge is chopped off inside Gaussian surface. In the figure below Q_{enclosed} is shown in blue.



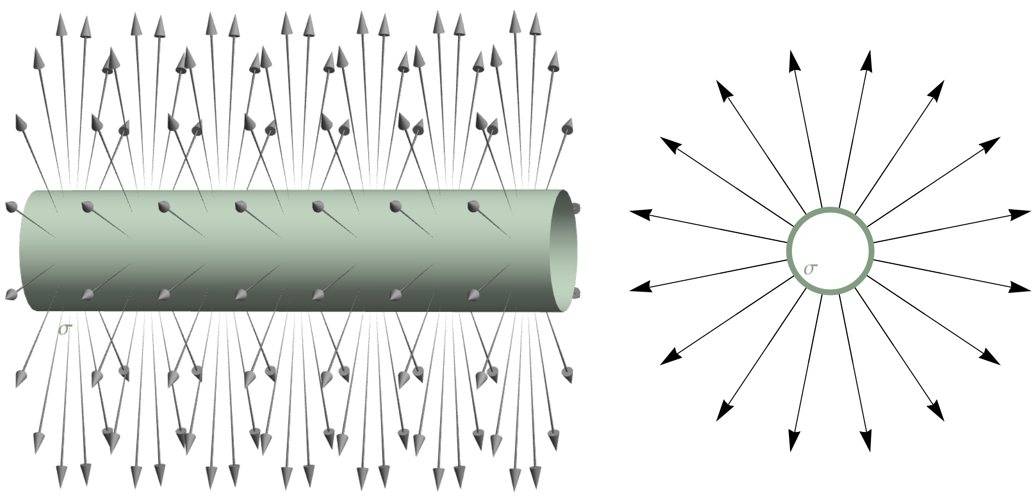
Q_{enclosed} in blue for the $r > R$ case (left) and its cross-section (right)

This is a cylindrical shell of radius R, the radius of the charged tube, and length ℓ, the length of the Gaussian surface. The area of this is A_{enclosed} and the enclosed charge is $\sigma\,A_{\text{enclosed}}$

$$A_{\text{enclosed}} = 2\,\pi\,R\,\ell \implies Q_{\text{enclosed}} = \sigma\,A_{\text{enclosed}} = \sigma\,2\,\pi\,R\,\ell$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E_r\,2\,\pi\,r\,\ell = \frac{\sigma\,2\,\pi\,R\,\ell}{\varepsilon_0} \implies \vec{E} = E_r\,\hat{r} = \frac{\sigma\,R}{\varepsilon_0\,r}\,\hat{r}$$

Notice that the ℓ cancels, as it must.



Field lines for infinite thin-shelled cylinder (left) and its cross-section (right)

Planar Symmetry

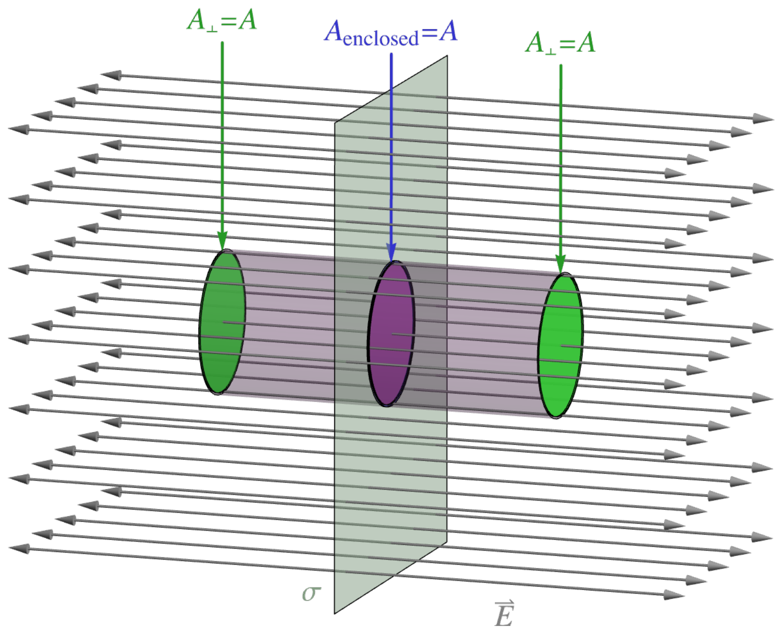
Here we will consider just one example. Consider an infinite uniform insulating plane of charge with surface charge density (charge/area) σ. The symmetry requires that the field is perpendicular to the plane. Drawing the field lines moving away from the line shows that the magnitude of the field must be uniform and thus independent of distance from the plane. If $\hat{n}$ is the unit vector pointing away form the plane then we get the field having the form

$$\vec{E} = E\,\hat{n}.$$

Take the Gaussian surface to be a right cylinder with faces of any cross sectional shape and area A with the faces on either side of the plane. The flat faces contribute a flux of $E\,A$ and the tube contributes zero flux. The charge enclosed by the surface is $Q_{\text{enclosed}} = \sigma\,A$

$$\oint \vec{E}\cdot d\vec{A} = \int_{\text{tube}} \vec{E}\cdot d\vec{A} + \int_{\text{end 1}} \vec{E}\cdot d\vec{A} + \int_{\text{end 2}} \vec{E}\cdot d\vec{A}$$
$$= 0 + E\,A + E\,A = E\,2\,A$$

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E\,2\,A = \frac{\sigma\,A}{\varepsilon_0} \implies \vec{E} = \frac{\sigma}{2\,\varepsilon_0}\,\hat{n}.$$

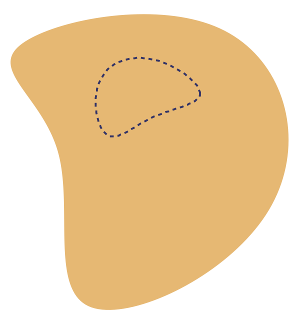


B.4 - Conductors in Electrostatics - I

Inside a conductor $\vec{E} = \vec{0}$.

Recall that a conductor has freely moving charges. If there is an electric field inside a conductor the free charges will move. This means that an electric field inside a conductor implies a current. Since currents are not allowed in electrostatics, it follows that the field inside a conductor must be zero.

There is no excess charge inside a conductor. All excess charge is on the surface of a conductor.



A Gaussian surface entirely inside a solid conductor.

Consider a Gaussian surface entirely inside a conductor. Since the electric field is zero, Gauss's law implies that $Q_{\text{enclosed}} = 0$. This means there is no excess charge in *any* region inside a conductor. There can be excess charge on a conductor, though; all excess charge is on the conductor's surface.

The electric field is perpendicular to the surface of a conductor and it is proportional to the surface charge density, $E = \sigma / \varepsilon_0$.

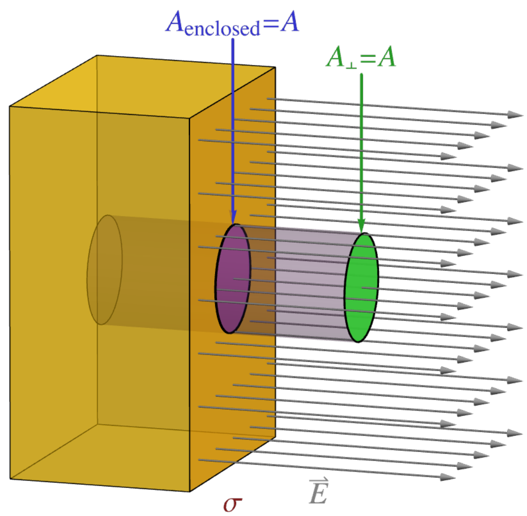
For the same reason that the field is zero inside a conductor, it must be perpendicular to the surface of a conductor. If there is a component parallel to the surface of a conductor then that parallel component will induce surface currents and this violates the assumptions of electrostatics.

Consider a small region on the surface of a conductor, that is sufficiently small compared to the curvature of the surface so that it can be considered flat. Imagine a cylindrical Gaussian surface with small flat faces parallel to the surface of the conductor, with one surface inside and one outside. Describe this as a tube and two ends as before. The tube is parallel to the field so the flux through it is zero. The field is zero inside the conductor, so the inside end gives zero flux. This leaves a flux of $E\,A$ at the outside end.

$$\oint \vec{E}\cdot d\vec{A} = \int_{\text{tube}} \vec{E}\cdot d\vec{A} + \int_{\text{inside end}} \vec{E}\cdot d\vec{A} + \int_{\text{outside end}} \vec{E}\cdot d\vec{A} = 0 + 0 + E\,A = E\,A$$

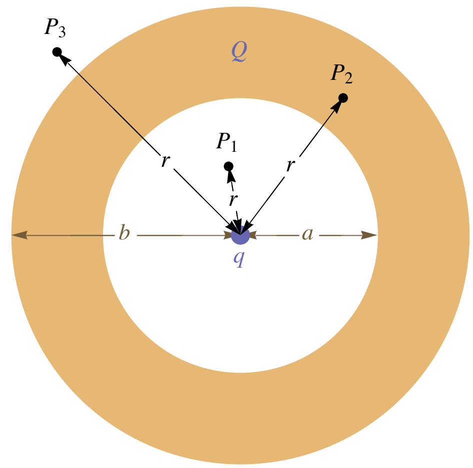
If the surface charge density at that position is σ, then the charge enclosed by the Gaussian surface is $Q_{\text{enclosed}} = \sigma\,A$. Define $\hat{n}$ as the outward unit normal at the surface of the conductor. Gauss's law gives the surface field as

$$\vec{E} = \frac{\sigma}{\varepsilon_0}\,\hat{n}.$$



Example B.6 - A Point Charge Inside a Hollow Conducting Sphere - Shielding

A hollow spherical conductor has concentric spherical surfaces with an inside radius of a and outside radius b. The conductor is given a net charge Q. At the conductor's center is a point charge q.

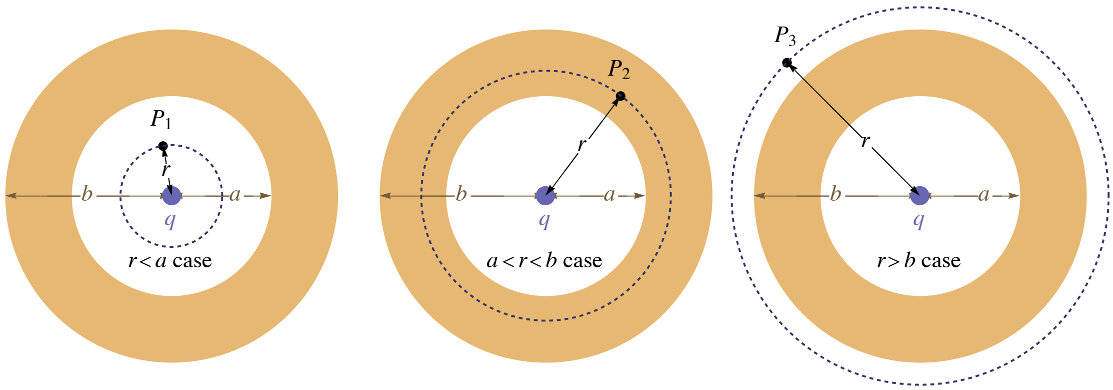


(a) What is the electric field as a function of r? Give answers for $r < a$, $a < r < b$ and $r > b$.

Solution

This is another example of Gauss's law with spherical symmetry.

$$\oint \vec{E}\cdot d\vec{A} = \frac{Q_{\text{enclosed}}}{\varepsilon_0} \implies E_r\,4\,\pi\,r^2 = \frac{1}{\varepsilon_0}\,Q_{\text{enclosed}} \implies \vec{E} = E_r\,\hat{r} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r}.$$



We now consider the three cases:

For $r < a$ we use the Gaussian surface through point P_1. The only charge inside this is the the point charge q.

$$Q_{\text{enclosed}} = q \implies \vec{E} = E_r\,\hat{r} = k_e\,\frac{q}{r^2}\,\hat{r}.$$

For $a < r < b$ the Gaussian surface is through point P_2. Here is logic is different; we are inside a conductor the field is zero.

$$\vec{E} = \vec{0}$$

For $r > b$ the Gaussian surface is outside the conductor through point P_3. The charge inside this Gaussian surface is the total charge, the point charge q and the conductor's charge Q.

$$Q_{\text{enclosed}} = q + Q \implies \vec{E} = E_r\,\hat{r} = k_e\,\frac{q+Q}{r^2}\,\hat{r}.$$

(b) Specify how the charge on the conductor is distributed by giving the charge on both surfaces.

Solution

We did not use Gauss's law when determining the field inside the conductor. Using Gauss's law will tell us how the conductor's charge is distributed. Because the field is zero the charge enclosed inside the Gaussian surface through P_2 must also vanish.

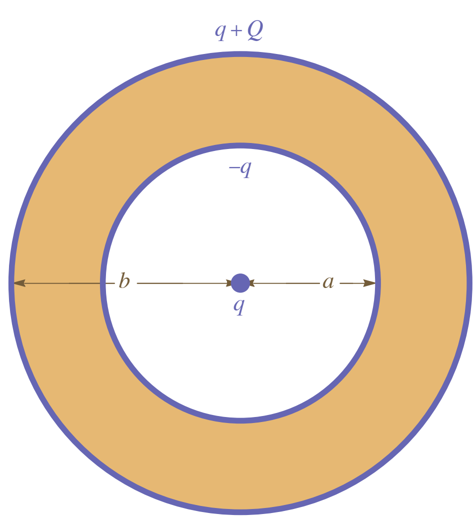
$$\vec{0} = \vec{E} = k_e\,\frac{Q_{\text{enclosed}}}{r^2}\,\hat{r} \implies Q_{\text{enclosed}} = 0$$

If there is no net charge inside the Gaussian surface through P_2 then the charge on the inside surface of the conductor must cancel the charge at the center.

$$Q(a) = -q$$

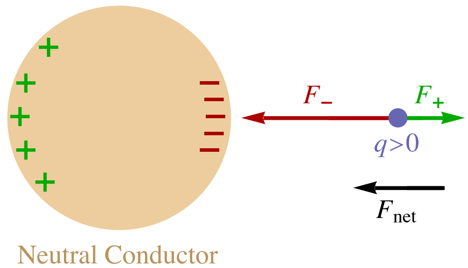
Since the total charge on the conductor is Q then the charge at the outside surface $Q(b)$ must combine with $Q(a)$ to give a total charge of Q.

$$Q = Q(a) + Q(b) \implies Q(b) = Q - Q(a) = Q - (-q) = q + Q$$



B.5 - Conductors and the Definition of $\vec{E}$

A point charge is always attracted to a neutral conductor. Consider a positive point charge q and a neutral conductor. The positive charge will polarize the conductor; this is, it will move charge in the conductor. Near q there will be a negative charge buildup and the far side of the conductor will have a positive charge build-up. Although there is as much positive as negative in the conductor, the negative is closer and will have a larger effect. This gives a net attractive force.



If we revisit the definition of the electric field, we see a problem. Let the charge q above be some test charge q_0. The effect of the test charge is to move around charge in conductors. This means that the presence of the test charge causes a change in the field and $\vec{E} = \vec{F} / q_0$ would depend on q_0. If we want to keep this definition of the field then we must insist that all charge is kept fixed in place, but with conductors this will not happen. Moving a test charge near a neutral conductor will create a field and thus the original definition will give a field where there is none.

The surface charge density induced on the conductor due to q_0 must be proportional to q_0; because of this the change in the force on q_0 due to the induced charges on the conductor is proportional to the q_0^2. We will modify the definition of $\vec{E}$, exploiting this q_0^2 proportionality. The modified definition is found by taking the limit.

$$\vec{E} = \lim_{q_0 \to 0} \frac{\vec{F}}{q_0}$$

This gives the correct field in the presence of conductors.