Chapter H

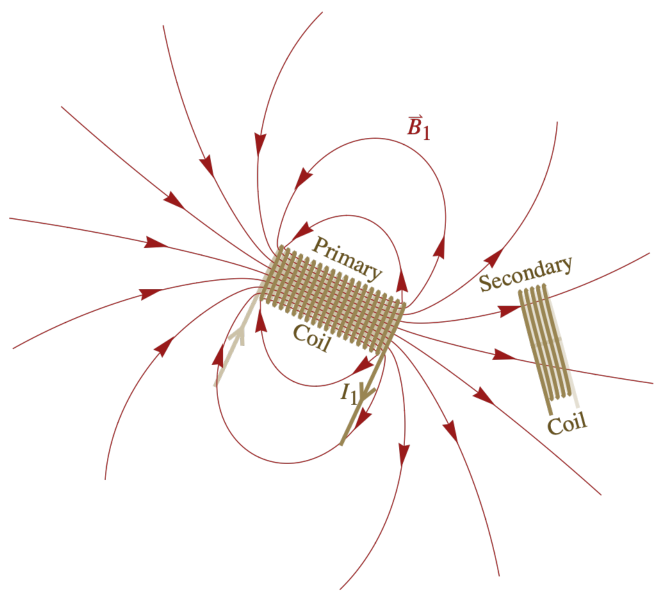
Inductance and Transient Circuits

Blinn College - Physics 2326 - Terry Honan

As a consequence of Faraday's law a changing current through one coil induces an EMF in another coil; this is known as mutual inductance. Similarly, a changing flux in a coil induces an EMF in the same coil; this is self inductance and a circuit component with inductance is called an inductor. We will also discuss simple circuits with inductors combined with our other linear circuit elements: resistors, capacitors and DC voltage sources.

H.1 - Mutual and Self Inductance

Mutual Inductance



Mutual Inductance

Consider a pair of coils, one called the primary coil and the other called the secondary. A current I_1 through the primary creates a magnetic field $\vec{B}_1$ which in turn creates a magnetic flux through the secondary coil Φ_{12}.

$$I_1 \implies \vec{B}_1 \implies \Phi_{12}$$

A changing current creates a changing flux which induces an EMF in the secondary coil.

$$\frac{d}{dt} I_1 \implies \frac{d}{dt} \Phi_{12} \implies \mathcal{E}_2$$

The above relationships are proportionalities. Define the constant of proportionality as the mutual inductance M_{12}.

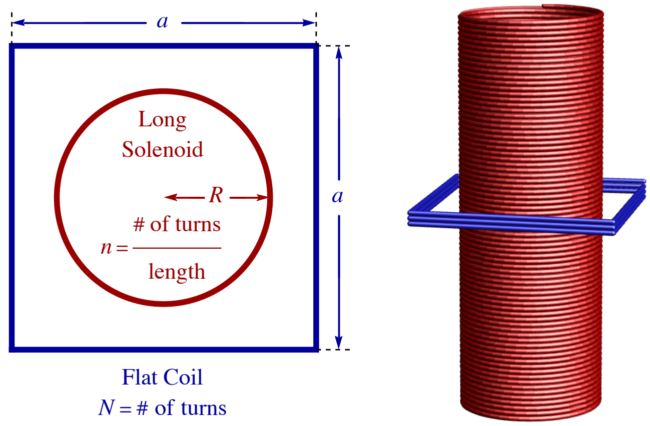
$$\mathcal{E}_2 = -M_{12} \frac{d}{dt} I_1$$

It is beyond the scope of this class to prove that the mutual inductance of coil 1 on coil 2 is the same as that of 2 on 1, and this will just be called M.

$$\mathcal{E}_2 = -M \frac{d}{dt} I_1 \text{ where } M = M_{12} = M_{21}$$

We will revisit mutual inductance in the next chapter in the context of a transformer. A transformer is the case where, ideally, all the flux from one coil passes through the other.

Example H.1 - Mutual Inductance



Top View (Left) and 3D Side View (Right)

A long solenoid with a vertical central axis has n turns per length and a radius R. this sits entirely inside a horizontal $a \times a$ square flat coil with N turns. What is the mutual inductance between these two coils?

Solution

In a mutual inductance problem, since $M_{12} = M_{21}$, we do not need to state whether we want to find M for the flat coil on the solenoid or for the solenoid on the flat coil. In the problem, since we know the field everywhere due to the solenoid, it is easy to find the mutual inductance for the solenoid on the flat coil; it is well beyond the scope of this course to find M for the flat coil on the solenoid. I_1 is the current in the solenoid and B_1 is the field due to I_1.

$$I_1 \implies B_1 = \begin{cases} \mu_0 \, n \, I_1 & \text{Inside} \\ 0 & \text{Outside} \end{cases} \implies \Phi_{12} = B_1 \, \pi \, R^2 = \mu_0 \, n \, I_1 \, \pi \, R^2$$

From this we can use Faraday's law to find the $\mathcal{E}_2$, the induced EMF in the flat coil

$$\frac{d}{dt} I_1 \implies \frac{d}{dt} \Phi_{12} \implies \mathcal{E}_2$$

$$\mathcal{E}_2 = -N \frac{d\Phi_{12}}{dt} = -N \, \mu_0 \, n \, \pi \, R^2 \frac{dI_1}{dt}$$

Using the definition of mutual inductance we can now read off M.

$$\mathcal{E}_2 = -M \frac{dI_1}{dt} \implies M = \mu_0 \, n \, N \, \pi \, R^2$$

Self Inductance

There is also inductance of a coil on itself. This is called self inductance. When we use the term inductance by itself self inductance is implied. A current through a coil creates a field and that causes a flux through the coil itself.

$$I \implies \vec{B} \implies \Phi$$

A changing current creates a changing flux with induces an EMF in the coil.

$$\frac{dI}{dt} \implies \frac{d}{dt} \Phi \implies \mathcal{E}$$

The above relationships are proportionalities. The constant of proportionality is defined as the inductance L.

$$\mathcal{E} = -L \frac{dI}{dt}$$

Units: The SI unit for Inductance, both mutual and self, is: H = henry = V · s / A

The sign in the above expression is due to Lenz's law. This is a back EMF, opposing the direction of current flow. Take ΔV to be the change in the voltage when moving through the inductor in the direction of the current. A simple Lenz's law analysis shows that if the current is increasing the voltage change is negative. If we write V as the voltage drop we get

$$\Delta V = -L \frac{dI}{dt} \text{ and } V = L \frac{dI}{dt}.$$

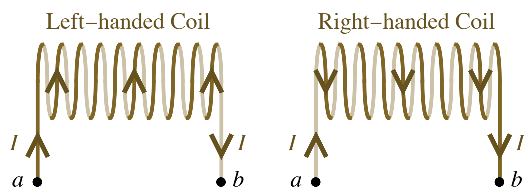
The sign conventions for inductors is the same as that for resistors and capacitors

$$\Delta V = -I R \text{ and } V = I R$$

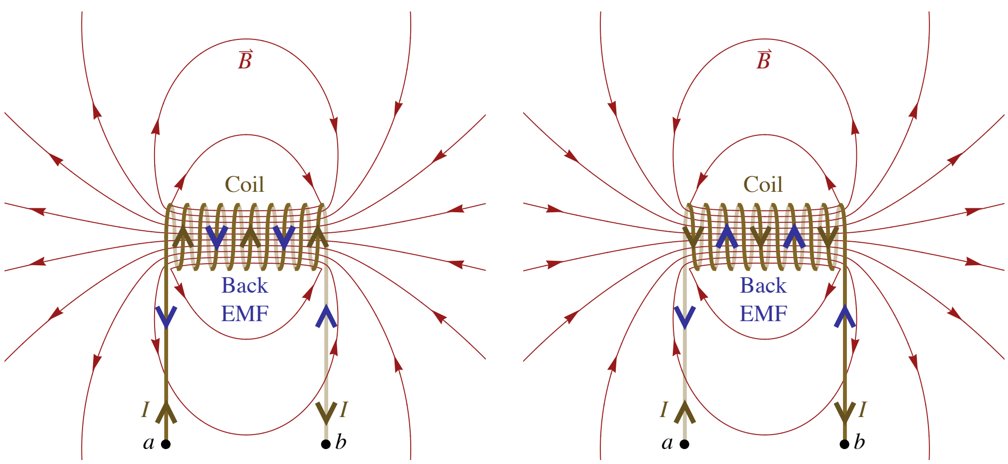
$$\Delta V = -\frac{Q}{C} \text{ and } V = \frac{Q}{C}$$

Example H.2 - Sign of ΔV for an Inductor

Show that regardless of how a coil is wrapped, the one on the left is left-handed and the one on the right is right-handed, the sign induced EMF is opposite the direction of the current flow. Take dI/dt to be positive and show that the voltage drops across the coil when passing through in the direction of the current.



Solution



The coil at the left is left-handed coil and at the right is right-handed. Regardless of how the coil is wrapped there is a back EMF. $\Delta V = V_b - V_a = -L\, dI/dt < 0$ when $dI/dt > 0$

For a left-handed coil if the current enters at a and exits at b then the magnetic field is to the left. Choosing the right normal to be positive the flux then is negative. With the current increasing then $d\Phi/dt$ is also negative. The induced flux is then positive and the resulting induced EMF is backward, from b to a.

Positive normal		Sign of Φ	Sign of $d\Phi/dt$	Sign of Φ_{ind}	Sense of $\mathcal{E}$ or I
→	Left–handed	−	−	+	from b to a
	Right–handed	+	+	−	from b to a

$$\Delta V = V_b - V_a = -L\, dI/dt < 0$$

Inductance of a Long Solenoid

Consider a long solenoid of length ℓ, with N turns and a cross-sectional area A. As before we define n as the number of turns per length $n = N/\ell$. Passing from the current to the field to the flux gives

$$I \implies B = \mu_0 \, n \, I \implies \Phi = B\,A = \mu_0 \, n \, I \, A$$

and using Faraday's law we get

$$\mathcal{E} = -N \frac{d\Phi}{dt} = -N \, \mu_0 \, n \, A \frac{dI}{dt}.$$

Using the definition of inductance $\mathcal{E} = -L \frac{dI}{dt}$, we can read the inductance from this expression. Writing L in terms of both N and n gives

$$L = \mu_0 \, n^2 \, A \, \ell = \mu_0 \frac{N^2}{\ell} A.$$

Example H.3 - Long Solenoid as an Inductor

(a) A long solenoid has 120 turns, a length of 75 cm and a 2 cm radius. What is its inductance?

Solution

$$N = 120, \; \ell = 0.75 \text{ m} \text{ and } r = 0.02 \text{ m} \implies A = \pi \, r^2 = 0.0012566 \text{ m}^2$$

Given the values above, we can find L.

$$\mu_0 = 4\,\pi \times 10^{-7} \frac{\text{N}}{\text{A}^2} \implies L = \mu_0 \frac{N^2}{\ell} A = 3.03 \times 10^{-5} \text{ H}$$

(b) A current given by

$$I(t) = 5\,t^2 - 8\,t + 12 \quad \text{(in SI Units)}$$

flows through this solenoid. What is the voltage drop across solenoid at $t = 2$ s?

Solution

$$\frac{dI}{dt} = 10\,t - 8 \implies \frac{dI}{dt} = \frac{dI}{dt}(2 \text{ s}) = 12 \text{ A/s}$$

This gives the voltage drop.

$$V = L \frac{dI}{dt} = 0.364 \text{ mV}$$

H.2 - Energy Considerations

Energy in an Inductor

Inductors, like capacitors, store energy, while resistors dissipate energy. Use U to denote the energy in an inductor. The rate that energy is being stored in an inductor is

$$\frac{dU}{dt} = \mathcal{P} = I\,V = I\,L \frac{dI}{dt}.$$

Integrating the above expression gives

$$U = \frac{1}{2} L \, I^2 + \text{constant}.$$

Choosing $U = 0$ when $I = 0$ fixes the constant and gives the desired expression.

$$U = \frac{1}{2} L I^2$$

Example H.4 - Long Solenoid as an Inductor (Continued)

(c) Given the solenoid and current described in parts (a) and (b), what is energy stored in the solenoid at $t = 2$ s?

Solution

$$I(t) = 5\,t^2 - 8\,t + 12 \quad \Longrightarrow \quad I = I(2\text{ s}) = 16\text{ A}$$

The energy can easily be calculated.

$$U = \frac{1}{2} L I^2 = 3.88 \times 10^{-3}\text{ J}$$

Energy in a Magnetic Field

In the capacitance chapter we derived an expression for the energy density (Energy/Volume) in an electric field.

$$u = \frac{1}{2} \varepsilon_0 E^2$$

To derive this we used the fact that the electric field is uniform inside a parallel plate capacitor. Combining expressions for the energy in a capacitor and for the capacitance gave the above expression for u. A similar analysis will give the energy density in a magnetic field.

The magnetic field is uniform inside a long solenoid. Combining the expression for the inductance of a long solenoid with the energy in an inductor gives

$$U = \frac{1}{2} L I^2 \text{ and } L = \mu_0 n^2 A \ell \quad \Longrightarrow \quad U = \frac{1}{2} \mu_0 n^2 A \ell I^2$$

Using $B = \mu_0 n I$, solve for I and insert.

$$U = \frac{1}{2} \mu_0 n^2 A \ell I^2 = \frac{1}{2} \mu_0 n^2 A \ell \left(\frac{B}{\mu_0 n} \right)^2 = \frac{1}{2 \mu_0} B^2 A \ell$$

$u = U / \text{Volume} = U / (A \ell)$ gives the energy density in a magnetic field.

$$u = \frac{1}{2 \mu_0} B^2$$

Example H.5 - Energy Density in the Earth's Field

The earth's magnetic field at Bryan Texas is 47.3 μT. What is the energy density (energy/volume) stored in this field?

Solution

$$\mu_0 = 4 \pi \times 10^{-7} \frac{\text{N}}{\text{A}^2} \text{ and } B = 47.3 \times 10^{-6}\text{ T} \quad \Longrightarrow \quad u = \frac{B^2}{2 \mu_0} = 8.90 \times 10^{-4} \frac{\text{J}}{\text{m}^3}$$

H.3 - Transient Circuits and Differential Equations

The voltage drops across a resistor, capacitor and inductor are, respectively,

$$V = I R, \quad V = \frac{Q}{C} \text{ and } V = L \frac{dI}{dt}.$$

Remember that these are voltage drops, so $\Delta V = -V$. If we build a circuit out of these and dc voltage sources, where $\Delta V = \mathcal{E}$, we then get an equation for I and Q. Since I is the time derivative of Q

$$I = \frac{dQ}{dt}$$

this will give a differential equation for Q or for I.

Comments on ODEs (Ordinary Differential Equations)

- A differential equation (DE) is some equation involving a function and its derivatives.

  The differential equation is solved to find the function.

- The order of a DE is the highest number of derivatives.

  If there is at most a second derivative it is a second order equation.

- If it is a function of one variable it is an ordinary differential equation (ODE). For functions of several variables there are partial differential equations (PDE).

  For functions of more than one variable we take partial derivatives instead of ordinary ones. The differential equations course (taken after Cal. III) is on ODEs.

- The general solution of a p^{th} order ODE is any solution involving p independent arbitrary constants.

  It is easy to verify that something is a solution to a differential equation; it is just a matter of taking derivatives and plugging into an equation. If a solution has the correct number of arbitrary constants then we can conclude that this is the general solution.

  As an example of this consider the simple example of a second order ODE from Physics I, motion with constant acceleration. Since this is a second order equation we will have two arbitrary constants.

$$\frac{d^2 x}{dt^2} = a = \text{const.} \implies \frac{dx}{dt} = v(t) = a\,t + C_1 \implies x(t) = \frac{1}{2}\,a\,t^2 + C_1\,t + C_2$$

  We can then interpret the arbitrary constants in terms of the initial velocity v_0 and initial position x_0.

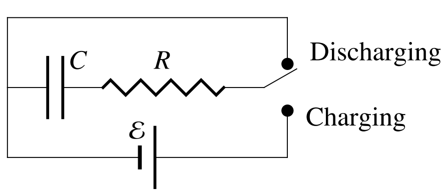
$$v_0 = v(0) \implies C_1 = v_0 \;\text{ and }\; x(0) = x_0 \implies C_2 = x_0 \implies x(t) = \frac{1}{2}\,a\,t^2 + v_0\,t + x_0$$

H.4 - RC Circuits

When the switch is thrown to the Charging position the current flows from the battery to charge the capacitor. In the Discharging position the charge flows from the capacitor and its energy is dissipated in the resistor. The charge on the capacitor is related to the current in the wire by

$$I = \frac{dQ}{dt}.$$

Note that when the capacitor is discharging the charge is decreasing and thus the current is negative.



Discharging

For the discharging case, applying the loop rule to the circuit gives:

$$0 = I\,R + \frac{Q}{C}.$$

Using the fact that the current is the derivative of the charge we can rewrite this as a first order differential equation.

$$0 = R\frac{dQ}{dt} + \frac{1}{C}Q \quad \text{or} \quad \frac{dQ}{dt} = -\frac{1}{\tau}Q$$

where τ is defined as the time constant

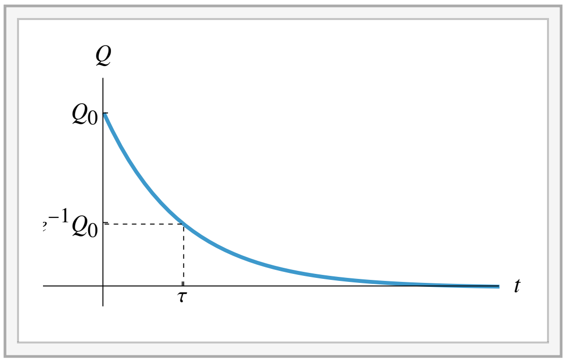
$$\tau = R\,C.$$

Since the derivative of an exponential is itself we can guess a solution of $e^{-t/\tau}$ and it is easy to verify that this is a solution. If we multiply this by a constant α then it still is a solution

$$Q(t) = \alpha\, e^{-t/\tau}.$$

Since α is arbitrary and we are beginning with a first order ODE we can conclude that this is the general solution. If we define the initial charge as Q_0 then $Q_0 = Q(0) = \alpha$ and the solution becomes

$$Q(t) = Q_0\, e^{-t/\tau}.$$



Interactive Figure - Discharging an RC Circuit

Charging

The loop rule for the charging case gives:

$$0 = \mathcal{E} - I\,R - \frac{Q}{C}$$

and we can rewrite this as a first order differential equation.

$$\mathcal{E} = R\frac{dQ}{dt} + \frac{1}{C}Q$$

Using the same time constant we can guess a solution of the form

$$Q(t) = \alpha\, e^{-t/\tau} + \beta.$$

Inserting this into our differential equation gives

$$\mathcal{E} = -R\frac{\alpha}{\tau}e^{-t/\tau} + \frac{1}{C}\left(\alpha\, e^{-t/\tau} + \beta\right)$$

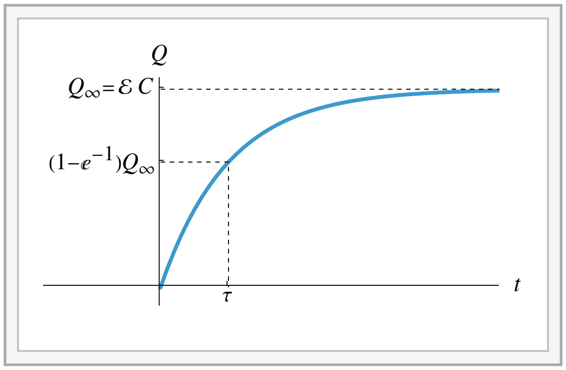
The terms involving α cancel and the equation requires that β has the value $\beta = \mathcal{E}\,C$ for it to be a solution. This means that

$$Q(t) = \alpha\, e^{-t/\tau} + \mathcal{E}\,C$$

is a solution to the differential equation. Since it is a first order equation and α is an arbitrary constant we can conclude that it is the general solution.

The solution we desire is where at $t = 0$ the capacitor is uncharged $Q(0) = 0$, giving

$$Q(t) = Q_\infty\left(1 - e^{-t/\tau}\right) \text{ where } Q_\infty = \mathcal{E}\,C.$$



Interactive Figure - Charging an RC Circuit

Transient behavior in General RC Circuits.

We can make some general comments about the time-dependent behavior of general circuits, not just series circuit, that involve resistors and capacitors.

- When a capacitor has no charge the voltage across it is zero. Current flows to it as if the capacitor is not there at all and it behaves like it is shorted out; it is equivalent to replacing it with a wire with zero resistance, $R = 0$.
- After a long time any capacitor will be fully charged and no current will flow to it. It now behaves as a break in the circuit and it is equivalent to an infinite resistance, $R \to \infty$

Example H.6 - How Many Time Constants?

How many time constants are required for a charging capacitor to reach 99.999% its maximum charge.

Solution

Start with the formula for a charging capacitor.

$$Q(t) = Q_\infty \left(1 - e^{-t/\tau}\right)$$

We are given the ratio of Q to Q_∞.

$$0.99999 = \frac{Q}{Q_\infty} = 1 - e^{-t/\tau} \implies e^{-t/\tau} = 1 - 0.99999 = 0.00001$$

$$\implies t = -\tau \ln(0.00001) = 11.5\,\tau$$

Example H.7 - Discharging Capacitor

Using a 250 μF capacitor, a 300 Ω resistor and a 12 V battery, fully charge the capacitor across the battery. Disconnect the capacitor from the battery and connected it across the resistor at time $t = 0$.

(a) What is the initial charge on the capacitor? (at $t = 0$)

Solution

First list what we are given.

$$C = 250 \times 10^{-6}\,\text{F},\;\; R = 300\,\Omega \;\text{ and }\; V = \mathcal{E} = 12\,\text{V}$$

From the formula for a charging capacitor we see that the initial charge for the discharging is the Q_∞ from charging.

$$Q(t) = Q_\infty \left(1 - e^{-t/\tau}\right) \text{ where } Q_0 = Q_\infty = \mathcal{E}\,C = 0.003\,\text{C}$$

(b) At $t = 55$ ms, what are the charge on, the voltage across and the current through the capacitor?

Solution

$$t = 0.055\,\text{s} \;\text{ and }\; \tau = R\,C = 0.075\,\text{s}$$

$$Q(t) = Q_0\, e^{-t/\tau} \implies Q(0.055\,\text{s}) = 1.44 \times 10^{-3}\,\text{C}$$

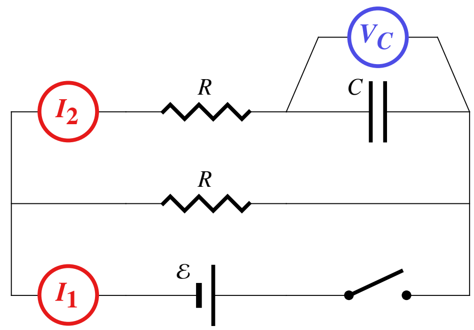
Now find the voltage.

$$V = Q/C = 5.76 \text{ V}$$

For the current as a function of time differentiate the charge.

$$I(t) = \frac{dQ}{dt} = -\frac{Q_0}{\tau} e^{-t/\tau} \implies I(0.055 \text{ s}) = -0.0192 \text{ A}$$

Example H.8 - General Transient Behavior



(a) Just after the switch is closed, what are the two currents through the ammeters I_1 and I_2, and what is the voltage across the capacitor, V_C.

Solution

The voltage across a fully discharged capacitor is zero.

$$V_C = 0$$

When the capacitor is fully discharged it behaves as a short in the circuit. Imagine replacing it with a wire. The voltage source sees itself connected across to equal resistances R in parallel, so $R_{\text{eq}} = R/2$.

$$I_1 = \mathcal{E} \Big/ R_{\text{eq}} = 2\,\mathcal{E}/R$$

Since the voltage across the capacitor is zero with zero charge, the current through the resistor in series with it sees the full voltage of the source $\mathcal{E}$.

$$I_2 = \mathcal{E}/R$$

(b) A long time after the switch is closed, what are the two currents through the ammeters I_1 and I_2, and what is the voltage across the capacitor, V_C.

Solution

Now the capacitor behaves a a break in the circuit and the current to it vanishes.

$$I_2 = 0$$

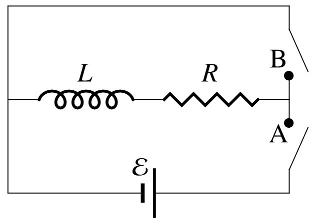
With I_2 being zero there is no voltage drop across the resistor in series with the capacitor and the capacitor sees the full source voltage.

$$V_C = \mathcal{E}$$

The voltage source sees itself connected across just one resistances R.

$$I_1 = \mathcal{E}/R$$

H.5 - RL Circuits



Decaying Current

Begin with switch **A** closed and **B** opened. This creates a current through the inductor and resistor. Close switch **B** and then open **A**. This causes the current to flow through the top branch of the above circuit. Applying the loop rule around the circuit gives

$$0 = L \frac{dI}{dt} + R\,I.$$

This is a first order ODE (ordinary differential equation) similar to that of a discharging capacitor

$$\frac{dI}{dt} = -\frac{1}{\tau}\,I,$$

where the time constant τ is defined by

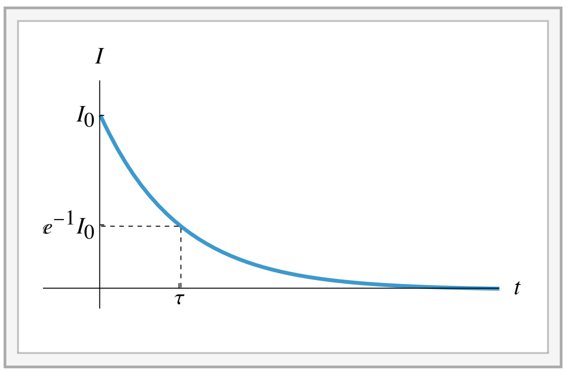
$$\tau = \frac{L}{R}.$$

The general solution of this differential equation for the current $I(t)$ is

$$I(t) = I_0\,e^{-t/\tau},$$

where the arbitrary constant is labelled I_0 because it is the current at time zero. This is a simple exponential decay, analogous to the decay of the charge for a discharging capacitor.

The initial energy in the inductor $U = \frac{1}{2}\,L\,I_0^2$ is converted to heat in the resistor.



Interactive Figure - Current Decay in an RL Circuit

Growing Current

With both switches opened giving zero current, close switch **A** at $t = 0$. This causes the current to gradually build up to a steady-state value. Apply the loop rule to the circuit gives the first order ODE.

$$\mathcal{E} = L \frac{dI}{dt} + R\,I.$$

Guess a solution of the form

$$I(t) = \alpha\,e^{-t/\tau} + \beta.$$

Inserting this guess into the differential equation gives

$$\mathcal{E} = L\left(-\frac{\alpha}{\tau}\,e^{-t/\tau}\right) + R\left(\alpha\,e^{-t/\tau} + \beta\right).$$

The definition of the time constant gives a cancellation of the $e^{-t/\tau}$ terms for any value of α, making α our arbitrary constant. Our guess will only be a solution when β has the value

$$\beta = \frac{\mathcal{E}}{R}.$$

Since we have a solution with one arbitrary constant and it is a first order ODE we can conclude that

$$I(t) = \alpha\, e^{-t/\tau} + \frac{\mathcal{E}}{R}$$

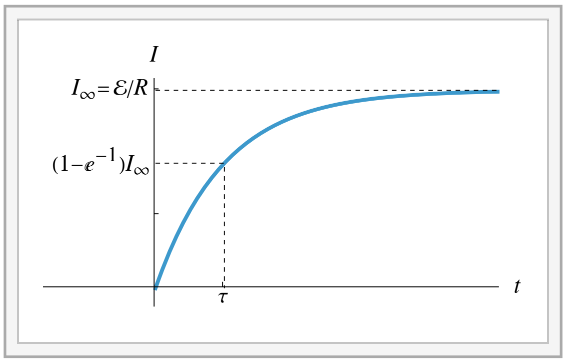
is the general solution. We are looking for a solution where the initial current is zero $I(0) = 0$, giving

$$\alpha = -\frac{\mathcal{E}}{R}.$$

The growth of the current can be written as

$$I(t) = I_\infty \left(1 - e^{-t/\tau}\right) \text{ where } I_\infty = \frac{\mathcal{E}}{R}$$

is the steady-state current.



Interactive Figure - Current Growth in an RL Circuit

Example H.9 - LR Current Decay

A 60 mH inductor is connected across an unknown resistance. Suppose initially there is a current around this circuit. If the current drops to one-fifth its value in 25 ms then what is the unknown resistance? The circuit contains only the inductor and resistor.

Solution

First, list what is given.

$$L = 0.060\text{ H},\quad t = 0.025\text{ s} \quad \text{and} \quad I/I_0 = 1/5$$

We can then solve to the time constant τ

$$I(t) = I_0\, e^{-t/\tau} \implies e^{-t/\tau} = \frac{I}{I_0} \implies \frac{t}{\tau} = -\ln\left(\frac{I}{I_0}\right) \implies \tau = -\frac{t}{\ln(I/I_0)} = 0.015533\text{ s}$$

and then the resistance.

$$\tau = L/R \implies R = L/\tau = \frac{0.060\text{ H}}{\tau} = 3.86\,\Omega$$

Example H.10 - LR Current Growth

An inductor with inductance L, a resistor with resistance R are connected across an EMF $\mathcal{E}$ in a simple series loop circuit.

(a) What are the current through the circuit, the voltage across the inductor and the voltage across the resistor just after the circuit is connected?

Solution

$$I(t) = \frac{\mathcal{E}}{R}\left(1 - e^{-t/\tau}\right)$$

It follows that at time zero, the current is zero, the voltage across the resistor is zero and the full EMF is across the inductor.

$$I = 0, \;\; V_R = 0 \;\; \text{and} \;\; \mathcal{E} = V_L + V_R \implies V_L = \mathcal{E}$$

(b) What are the current through the circuit, the voltage across the inductor and the voltage across the resistor a long (infinite) time after the circuit is connected?

Solution

After a long time the current will approach a steady-state where $dI/dt = 0$. With that

$$V_L = L\frac{dI}{dt} = 0.$$

It follows then that the voltage across the resistor is the EMF. Ohm's law then gives the current through the resistor which is the same, since it is a series circuit, as the current through the inductor.

$$\mathcal{E} = V_L + V_R \implies V_R = \mathcal{E} \;\; \text{and} \;\; I = V_R/R = \mathcal{E}/R$$

(c) After two time constants, what are the current through the circuit, the voltage across the inductor and the voltage across the resistor ?

Solution

The formula for current growth gives the current.

$$t = 2\tau \implies I = \left(1 - e^{-2}\right)\frac{\mathcal{E}}{R} = 0.864\,\frac{\mathcal{E}}{R}$$

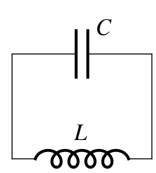
Ohm's law gives V_R and from that we can find V_L.

$$V_R = I\,R = \left(1 - e^{-2}\right)\mathcal{E} = 0.864\,\mathcal{E} \;\; \text{and} \;\; \mathcal{E} = V_L + V_R \implies V_L = \mathcal{E} - V_R = e^{-2}\,\mathcal{E} = 0.135\,\mathcal{E}$$

H.6 - LC Circuits, RLC Circuits and Their Mechanical Equivalents

There is an important analogy between the LC Circuit and the mass-spring system. We will see that the solution to both equations is oscillatory. We can add damping to these cases by inserting a resistor into the electrical system and adding friction to the mechanical system.

The LC Circuit



Consider a simple loop circuit containing just a capacitor C and inductor L. The loop rule gives

$$0 = L\frac{dI}{dt} + \frac{Q}{C}.$$

Write the current I as the time derivative the charge on the capacitor

$$I = \frac{dQ}{dt}$$

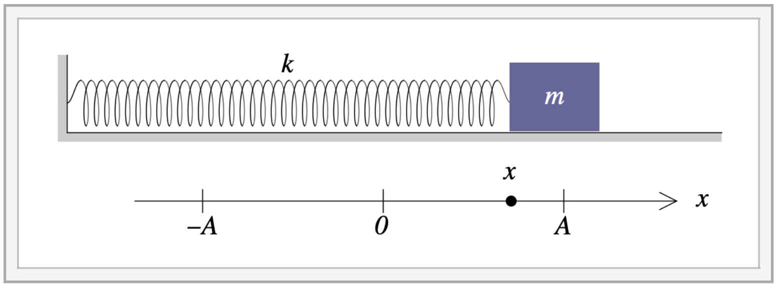
and define the angular frequency by

$$\omega_0 = \frac{1}{\sqrt{LC}}.$$

This gives the second order ODE

$$0 = \frac{d^2 Q}{d t^2} + \omega_0^2 Q .$$

The Mass-Spring System



Interactive Figure

The mechanical analog of this is a mass-spring system. The force of a spring is given by Hooke's law $F = -k x$. Applying Newton's second law gives a second order ODE

$$F_{\text{net}} = m a \implies -k x = m \frac{d^2 x}{d t^2}$$

Defining the angular frequency by

$$\omega_0 = \sqrt{\frac{k}{m}} ,$$

gives the second order ODE

$$0 = \frac{d^2 x}{d t^2} + \omega_0^2 x .$$

The Analogy without Damping

Summarizing the analogy as stated above: The charge, current and derivative of the current are the analogs of the position, velocity and acceleration. The energy in the inductor $U_L = \frac{1}{2} L I^2$ and the kinetic energy of the mass $K = \frac{1}{2} m v^2$ are analogous, as are the energy in the capacitor $U_C = \frac{1}{2C} Q^2$ and the potential energy of the spring $U = \frac{1}{2} k x^2$.

LC Circuit	Mass–Spring
Q	x
$I = \frac{dQ}{dt}$	$v = \frac{dx}{dt}$
$\frac{dI}{dt}$	$a = \frac{d^2x}{dt^2}$
L	m
C	$\frac{1}{k}$
$\omega_0 = \frac{1}{\sqrt{LC}}$	$\omega_0 = \sqrt{\frac{k}{m}}$
$U_L = \frac{1}{2} L I^2$	$K = \frac{1}{2} m v^2$
$U_C = \frac{1}{2C} Q^2$	$U = \frac{1}{2} k x^2$

Solution of the Undamped ODE

With the analogy clearly stated, let us solve the second order ordinary differential equation

$$0 = \frac{d^2 Q}{d t^2} + \omega_0^2 Q \,.$$

Since the second derivatives of both sine and cosine are the negatives of themselves, it follows that

$$\cos(\omega_0\, t) \text{ and } \sin(\omega_0\, t)$$

are solutions to our differential equation. Because the ODE has the simple properties (a homogeneous linear equation) that:

(i) a constant times a solution is a solution and
(ii) the sum of two solutions is a solution,

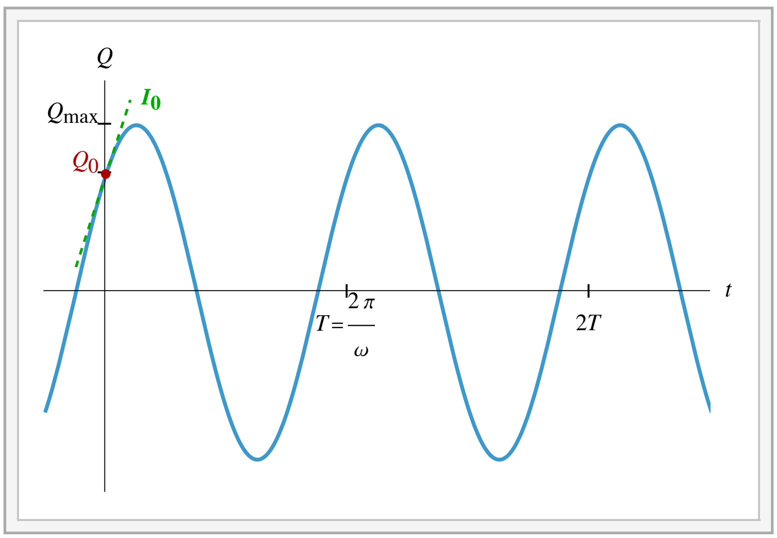
it follows that

$$Q(t) = B\cos(\omega_0\, t) + C\sin(\omega_0\, t)$$

is a solution, where B and C are arbitrary constants. Since we have solution to a second order ODE with two arbitrary constants, we can conclude that this is the general solution. Setting $Q(0) = Q_0$ and $I(0) = I_0$ where $I = dQ/dt$ gives $B = Q_0$ and $C = I_0/\omega_0$. Another way of presenting this solution is

$$Q(t) = Q_{\max}\cos(\omega_0\, t + \phi)$$

where the arbitrary constants are $Q_{\max}$, the amplitude, and ϕ, the phase angle.



Interactive Figure - Charge as a Function of Time for an LC Circuit

Example H.11 - LC Circuit

A 250 μF capacitor is given an initial charge of 30 μC before being connected across a 60 mH inductor. What are the charge on the capacitor and current through the inductor 4.2 ms after the circuit is connected?

Solution

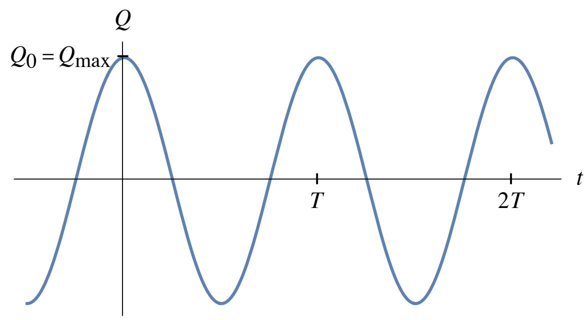
$$C = 250\times10^{-6}\,\text{F},\;\; Q_0 = 30\times10^{-6}\,\text{C},\;\; L = 0.060\,\text{H} \text{ and } t = 4.2\text{ ms} = 0.0042\text{ s}$$

First, we need to find the angular frequency.

$$\omega_0 = \frac{1}{\sqrt{L\,C}} = 365.15\text{ s}^{-1}$$

Because the capacitor is initially at its maximum charge $Q_0 = Q_{\max}$, the phase constant ϕ must be zero.

$$Q(t) = Q_{\max}\cos(\omega_0\, t + \phi) \implies Q(t) = Q_0\cos(\omega_0\, t)$$



To find the charge we just plug in the time.

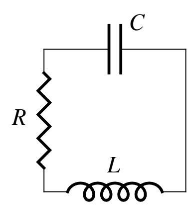
$$Q(0.0042\text{ s}) = Q_0 \cos(\omega_0\, t) = 1.11 \times 10^{-6}\text{ C}$$

To find the current we differentiate and then plug in the time.

$$I(t) = \frac{dQ}{dt} = -\omega_0\, Q_0 \sin(\omega_0\, t) \implies I(0.0042\text{ s}) = -0.0109\text{ A}$$

It is important to note that to calculate the above values you must have your calculator set in the radians mode.

LCR Circuit



We can add damping to the LC circuit by adding a resistor R to the series circuit. The loop rule gives

$$0 = L\,\frac{dI}{dt} + I\,R + \frac{Q}{C}.$$

This gives us the second order ODE

$$0 = \frac{d^2 Q}{dt^2} + \beta\,\frac{dQ}{dt} + \omega_0^2\, Q$$

where ω_0 is the same as in the undamped case and

$$\beta = \frac{R}{L}.$$

The Damped Mass-Spring System

The mechanical damping is achieved by adding viscous friction, which is the force $-bv$, to the Hooke's law force. Newton's second law gives

$$F_{\text{net}} = m\,a \implies -k\,x - b\,v = m\,\frac{d^2 x}{dt^2}.$$

This becomes the second order ODE

$$0 = \frac{d^2 x}{dt^2} + \beta\,\frac{dx}{dt} + \omega_0^2\, x$$

where ω_0 is the same as in the undamped case and now

$$\beta = \frac{b}{m}.$$

The Analogy with Damping

We can add to the previous table the values of the damping constants β.

LCR Circuit	Damped Mass–Spring
R	b
$\beta = \frac{R}{L}$	$\beta = \frac{b}{m}$

Note the similarities between resistance and friction. Both remove energy from the system, friction removes mechanical energy and resistance removes electrical energy. The energy lost to each goes to heat.

Solution with Damping

$$0 = \frac{d^2 Q}{d t^2} + \beta \frac{d Q}{d t} + \omega_0^2 Q$$

To solve the above differential equation we will guess a solution of the form

$$Q(t) = A\, e^{-\gamma t} \cos(\omega t + \phi).$$

This will be a solution only when the constants γ and ω have specific values, which we will write in terms of the constants in the equation, β and ω_0. The constants A and ϕ will be our arbitrary constants. To verify this is a solution and find γ and ω, we must plug our guess into the differential equation. First we must evaluate the derivatives

$$Q = A\, e^{-\gamma t} \cos(\omega t + \phi).$$

$$\frac{d Q}{d t} = A\, e^{-\gamma t} [-\gamma \cos(\omega t + \phi) - \omega \sin(\omega t + \phi)].$$

$$\frac{d^2 Q}{d t^2} = A\, e^{-\gamma t} \left[\left(\gamma^2 - \omega^2\right) \cos(\omega t + \phi) + 2 \gamma \omega \sin(\omega t + \phi)\right].$$

$$0 = \frac{d^2 Q}{d t^2} + \beta \frac{d Q}{d t} + \omega_0^2 Q \implies$$

$$\begin{aligned}
0 = \; & A\, e^{-\gamma t} \left[\left(\gamma^2 - \omega^2\right) \cos(\omega t + \phi) \; + \; 2 \gamma \omega \sin(\omega t + \phi)\right] \\
& + A\, e^{-\gamma t} [\; -\beta \gamma \; \cos(\omega t + \phi) \; - \; \beta \omega \; \sin(\omega t + \phi)] \\
& + A\, e^{-\gamma t} [\; \omega_0^2 \; \cos(\omega t + \phi) \; + \; 0 \;]
\end{aligned}$$

For this equality to be true at all times, the terms multiplying $\cos(\omega t + \phi)$ and $\sin(\omega t + \phi)$ must separately vanish, giving

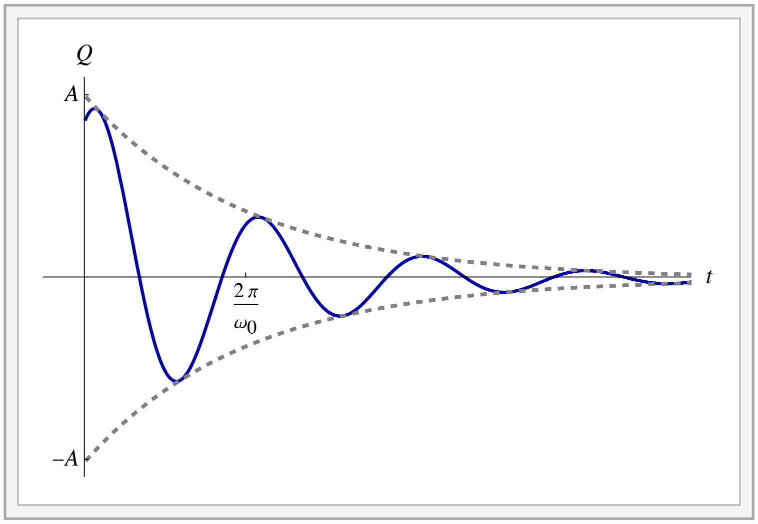
$$0 = \gamma^2 - \omega^2 - \beta \gamma + \omega_0^2 \text{ and } 0 = 2 \gamma \omega - \beta \omega.$$

The second expression gives

$$\gamma = \frac{\beta}{2}$$

and plugging this into the first expression gives

$$\omega = \sqrt{\omega_0^2 - \frac{\beta^2}{4}}\,.$$



Interactive Figure - Charge as a Function of Time for an LCR Circuit