Chapter K

Geometric Optics

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K.1 - Properties of Light

The Speed of Light

The speed of light in a vacuum is approximately $c \simeq 3.0 \times 10^8\ \mathrm{m/s}$. Because of its most fundamental nature we can choose our units to make c have the exact value

$$c = 2.99792458 \times 10^8\ \mathrm{m/s}.$$

The definition of a second is given in terms of our most accurate method of measuring time, with atomic clocks, and the above definition of c then gives a definition of the meter.

Historically, the first good estimate of the speed of light was made by Roemer. By plotting the period of Io, one of Jupiter's moons, Roemer was able to explain the lack of periodicity in Io's orbit in terms of the time difference for light from Io to reach the Earth as the relative distance between Io and Earth varies. An accurate measure of c can be found by shining a light through a rapidly rotating toothed wheel and reflecting it off a distant mirror.

The speed of light is a fundamental speed limit. It is impossible to send any information faster than c.

Example K.1 - Time of Travel for Light.

(a) The earth-sun distance is

$$R_{ES} = 1.50 \times 10^{11}\ \mathrm{m}$$

How long does it take for a light signal to travel from the sun to the earth?

Solution

Since $d = v\,t = c\,t$ we can simply solve for t.

$$t = \frac{d}{c} = \frac{R_{ES}}{c} = 500\ \mathrm{s} = 8.33\ \mathrm{min}$$

With the speed of light as the ultimate speed limit this means that we cannot know what is happening on the sun now, only what happened over eight minutes ago.

(b) The earth-moon distance is

$$R_{EM} = 3.84 \times 10^8\ \mathrm{m}$$

How long does it take for a light signal to travel from the moon to the earth?

Solution

$$t = \frac{d}{c} = \frac{R_{EM}}{c} = 1.28\ \mathrm{s}$$

There was a very noticeable time delay when astronauts were on the moon. It took over two and a half seconds for a radio signal to get from the earth to the moon and back.

(c) How far does light travel in a nano-second?

Solution

$$t = 10^{-9}\ \mathrm{s} \implies d = c\,t = 0.3\ \mathrm{m} \simeq 1\,\mathrm{foot}$$

We think of the speed of light as a significant communication obstacle when dealing with large distances, but with computers time scales are on the order of nano-seconds so the speed of light is also significant. As computers get faster, they must necessarily get smaller.

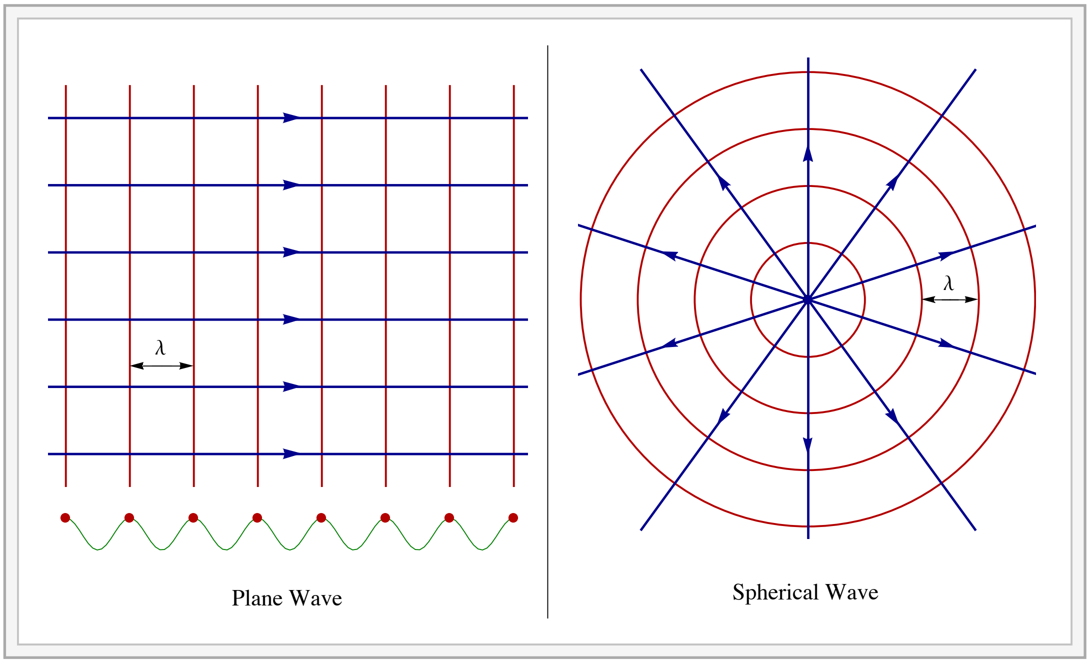
(d) A light-year ly is the distance light travels in one year. What is this in meters?

Solution

$$1\text{ ly} = c\times(1\text{ yr}) = 3.00\times10^{8}\ \frac{\text{m}}{\text{s}}\times 365.24\text{ days}\times\frac{24\text{ h}}{\text{day}}\times\frac{3600\text{ s}}{\text{h}} = 9.47\times10^{15}\text{ m}$$

Wave Fronts and Rays

There are two ways we can visualize waves, as wave fronts or as rays. If we mark the crest of each wave then these form the wave fronts. For a plane wave the crest corresponds to parallel planes moving at the wave speed perpendicular to the plane.



The plane wave correspond to our plane wave solution where the fields are constant along planes; these planes are the wave fronts. Light from a point source gives spherical waves. In both cases, the distance between wave fronts is a wavelength. When a point source is a long distance away the spherical wave can also be described as a plane wave. Light rays are what you would expect from the term; they are "beams" of light perpendicular to the wave front. The beam of a laser behaves as a ray.

Light in a Medium

When we refer to c as the speed of light it is implied that it is the speed in a vacuum. In a medium light slows down. It slows down by a factor called the index of refraction; this is a material dependent constant n defined by

$$v = \frac{c}{n},$$

where v is the speed in the medium. Clearly $n \geq 1$ and the equality applies to a vacuum.

Material	Index of Refraction – n
Vacuum	1 (exact)
Air (0 °C, 1 atm)	1.000277
Water	1.333
Crown Glass	1.52
Diamond	2.417

Polycarbonate	1.59

Some indices of refraction

When light hits an interface between indices n_1 and n_2 both the frequency and wavelength cannot remain unchanged, since

$$f\,\lambda = v = \frac{c}{n}.$$

The frequency is the same on both sides of the interface. This is easy to see. At the interface the incoming radiation is varying at some frequency and generally driving a system at some frequency induces oscillations at the same frequency. Given that the frequencies are equal

$$f = f_1 = f_2$$

the wavelengths must change. The wavelength is then proportional to the wave speed and we get

$$\frac{\lambda_2}{\lambda_1} = \frac{v_2}{v_1} = \frac{n_1}{n_2} \implies n_1\,\lambda_1 = n_2\,\lambda_2.$$

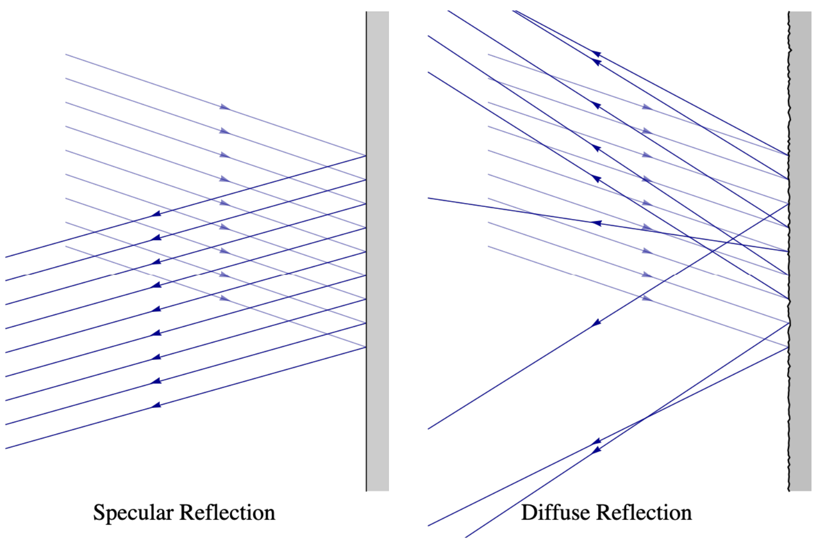
To view this a different way, if we define the vacuum wavelength as $\lambda_0 = c/f$ then the wavelength in a medium with index n is

$$\lambda = \frac{\lambda_0}{n}.$$

K.2 - Refraction and Reflection

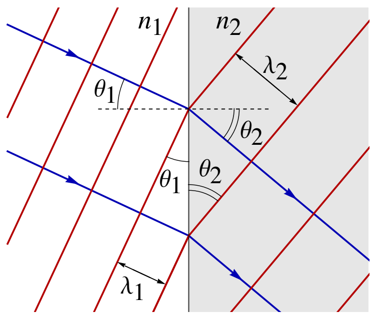
Reflection of Light

When we describe the angle that light makes with a surface, we measure the angle of the ray from the normal to the surface. When light reflects off a smooth shiny surface, it reflects so that the incident angle equals the reflected angle. This is known as the law of reflection. Most of the light we see is reflected light. Reflection off shiny smooth surfaces is called specular reflection. Most surfaces are rough on the order of the wavelengths of light. Because the normal to a rough surface is varying, the light then reflects in random directions. This is called diffuse reflection. Most of the light that we see is from the diffuse reflection off surfaces.

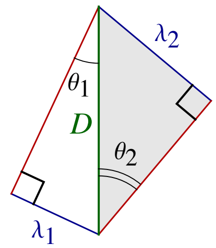


Refraction of Light and Snell's Law

As a consequence of the speed of light changing across an interface light will bend as it hits the interface. This is called refraction. Consider light passing from one medium to another through a flat interface. Take it to move from medium 1 with index n_1 to medium 2 with index n_2. We will, as usual, measure the angles of the rays relative to the normal to the surface.



The diagram shows wave fronts and rays on either side of the interface. Since the rays are perpendicular to the wave fronts, the angle between the rays and the normal are the same as the angle between the wave fronts and the interface. Enlarging the triangles at the center



and labeling the common side as D gives

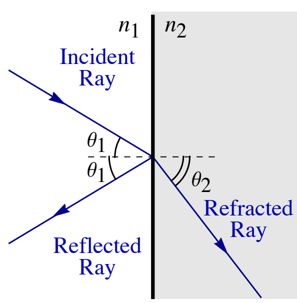
$$\sin\theta_1 = \frac{\lambda_1}{D} \text{ and } \sin\theta_2 = \frac{\lambda_2}{D} \implies \frac{\sin\theta_2}{\sin\theta_1} = \frac{\lambda_2}{\lambda_1} = \frac{n_1}{n_2}.$$

We can now write the law of refraction, called Snell's law, as

$$n_1 \sin\theta_1 = n_2 \sin\theta_2.$$

Incident, Reflected and Refracted Rays

In addition to refraction some of the light is reflected. The so-called law of reflection states that the angle, measured from the normal, of the reflected ray is the same as the incident.



Note that in this example the angle in medium 2 is larger than in medium 1. Since sine is an increasing function then, by Snell's law, the index in 2 is smaller than in 1.

$$\theta_1 < \theta_2 \implies \sin\theta_1 < \sin\theta_2 \implies n_1 > n_2$$

Example K.2 - Light from Air to Water

A laser beam with a wavelength of 560 nm is shot into water ($n_{water} = 1.333$) from air (take $n_{air} = 1$). Above the water the ray makes an angle of 35° from vertical. When inside the water:

(a) What is the angle of the ray from vertical?

Solution

This is a straight-forward application of Snell's law with

$$n_1 = 1, \quad n_2 = 1.333 \text{ and } \theta_1 = 35°.$$

We are looking for θ_2.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \implies \theta_2 = \sin^{-1}\left(\frac{n_1}{n_2}\sin\theta_1\right) = 25.5\,°$$

(b) What is the speed of the light?

Solution

The speed of light in a medium is $v = c/n$ where in the water the index is $n = n_{water}$.

$$c = 3.00\times 10^8\ \frac{\text{m}}{\text{s}} \text{ and } n = 1.333 \implies v = \frac{c}{n} = 2.25\times 10^8\ \frac{\text{m}}{\text{s}}$$

(c) What is the wavelength of the light?

Solution

The vacuum wavelength of the light is

$$\lambda_0 = 560\text{ nm} = 560\times 10^{-9}\text{ m}.$$

The wavelength in a medium is $\lambda = \lambda_0/n$.

$$\lambda = \lambda_0/n = 420\text{ nm} = 4.20\times 10^{-7}\text{ m}$$

(d) What is the frequency of the light?

Solution

When light moves from one medium to another its frequency does not change. The frequency of light in the water is the same as in the air.

$$f = f_0 = \frac{c}{\lambda_0} = 5.36\times 10^{14}\text{ Hz}$$

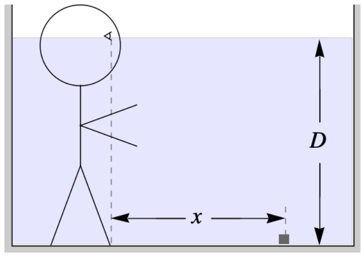
Total Internal Reflection

Suppose a light ray originates under water at an angle of θ_1 from the normal to the surface and emerging at an angle θ_2 in air. We will take the index for air to be one. How will the angle θ_2 change as we vary θ_1? The index of refraction for water is $n_{water} = 1.333$. Notice that θ_2 increases as θ_1 increases and is larger than θ_1. There is a point when the angle θ_2 cannot be found; this is when the calculation of $\sin^{-1}(n_{water}\sin(\theta_1))$ fails because you are evaluating the arcsin of an argument larger than one.

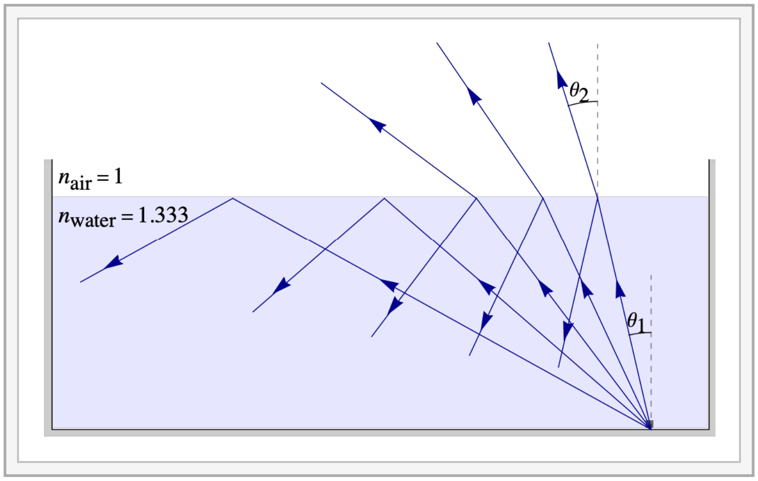
$$\theta_2 = \sin^{-1}(n_{water}\sin\theta_1)$$

θ_1	θ_2
10 °	13.38 °
20 °	27.12 °
30 °	41.80 °
40 °	58.96 °
45 °	70.49 °
48 °	82.14 °
49 °	−

Imagine standing in a pool of depth D with your eye just above the surface of water. If there is a small object on the floor of the pool a horizontal distance of x from your eye.



As you move backward to increase x there will be a point where the object disappears. What you see is the refracted ray and with a sufficiently large x that refracted ray will disappear. This is known as “total internal reflection”.



Consider the general case where light moves across an interface to a region with lower index.

$$n_1 > n_2$$

It follows that the angle from the normal increases ($\theta_1 < \theta_2$). Moreover, there is some angle where there is no refracted ray. In this case there is only the reflected ray. This is what we call total internal reflection. This occurs at incident angles above some critical angle

$$\theta_1 \geq \theta_{crit},$$

where the critical angle occurs when the refracted angle goes to 90°. That is, $\theta_1 = \theta_{crit}$ when $\sin\theta_2 = 1$, or

$$\sin\theta_{crit} = \frac{n_2}{n_1}.$$

Example K.3 - Critical Angle from Water to Air

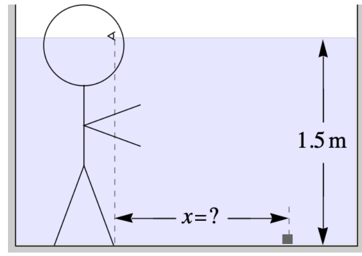
What is the critical angle for total internal reflection for light moving from water, with $n_{water} = 1.333$ to air?

Solution

$$\theta_{crit} = \sin^{-1}\left(\frac{n_2}{n_1}\right) = \sin^{-1}\left(\frac{1}{n_{water}}\right) = 48.6\,°$$

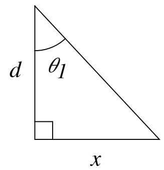
Example K.4 - An Object at the Bottom of a Pool

A kid stands in a 1.5-m deep pool with his eye just above the surface of calm water. What is the largest horizontal distance x that a small object can be from his eyes for him to be able to see it?



Solution

The light passes from the object to the surface and then refracts out to the eye. The incident angle on the water to air interface is given by



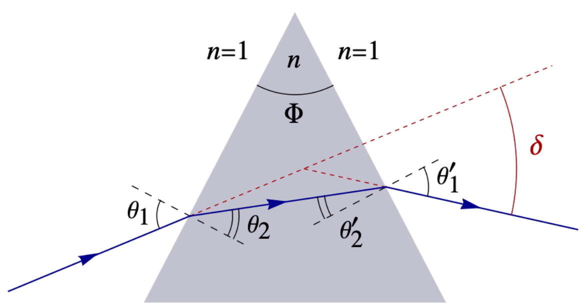
$$\Longrightarrow \quad \tan\theta_1 = \frac{x}{d} \quad \text{where} \quad d = 1.5\ \text{m}$$

The largest x is when $\theta_1 = \theta_{crit}$.

$$x = d\ \tan\theta_1 \quad \Longrightarrow \quad x_{max} = d\ \tan\theta_{crit} = d\ \tan 48.6\,° = 1.70\ \text{m}$$

Prisms and Dispersion

Deflection by a prism



Consider light hitting a prism as shown above. The apex angle Φ is the angle between the two refracting surfaces in the prism. If the ray incident on the prism is at an angle θ_1 from the normal it will refract to an angle θ_2, which can be found by Snell's law.

$$\sin\ \theta_1 = n\ \sin\ \theta_2$$

Now consider the triangle formed by the ray inside the prism and the top of the prism. Summing the internal angles of this triangle must give $180\,°$.

$$(90\,° - \theta_2) + (90\,° - \theta_2') + \Phi = 180\,° \quad \Longrightarrow \quad \theta_2 + \theta_2' = \Phi$$

The above expression allows us to find θ_2' from θ_2 and Φ. This lets us find θ_1'.

$$\sin\ \theta_1' = n\ \sin\ \theta_2'$$

The total angle of deflection δ is found by summing the bending at both interfaces. At the first the ray bends by $\theta_1 - \theta_2$ and at the second by $\theta_1' - \theta_2'$. It follows that δ is the sum.

$$\delta = (\theta_1 - \theta_2) + (\theta_1' - \theta_2') = \theta_1 + \theta_1' - \Phi$$

Dispersion

A prism splits light into the visible spectrum: red, orange, yellow, green, blue, indigo and violet. We have just seen that using Snell's law and knowing the prism's apex angle Φ and the incident angle θ_1, we can uniquely solve for the position of the ray leaving the prism in terms of the index of refraction of the prism. The fact that different colors refract to different angles implies that the index of refraction varies with color (wavelength). This is known as dispersion.

In a prism, red is the color that refracts the least; this means the index of red is the least. Red has the longest visible wavelength. It is generally the case that the index of refraction decreases with wavelength. For instance in the case of crown glass the tabulated index is 1.52; this is a mean value. As the wavelength increases from 400 to 700 nm, the index decreases from about 1.53 to 1.51.

Example K.5 - Angular Separation of Colors in a Prism

White light is incident at an angle of 15° on a prism with $\Phi = 25°$ of crown glass as shown in the prism diagram above. Using the expressions above, find the angular separation between the violet ray and red ray, where the indices are 1.53 for violet and 1.51 for red.

Solution

We are given the apex angle of the prism Φ and the incident angle for the white light we have $\theta_1 = 25°$, which applies to both colors. We also know the indices of refraction for both wavelengths.

$$\Phi = 25°,\ \ \theta_1 = 15°,\ \ n_{\text{red}} = 1.51 \text{ and } n_{\text{violet}} = 1.53$$

Using Snell's law at the first interface, $\sin\theta_1 = n\sin\theta_2$, we can find two angles θ_2.

$$\theta_{2,\text{red}} = \sin^{-1}\left(\frac{1}{n_{\text{red}}}\sin\theta_1\right) = 9.869° \text{ and } \theta_{2,\text{violet}} = \sin^{-1}\left(\frac{1}{n_{\text{violet}}}\sin\theta_1\right) = 9.739°$$

$\theta_2 + \theta_2' = \Phi$ relates the two angles inside the prism allowing us to find both θ_2'.

$$\theta'_{2,\text{red}} = \Phi - \theta_{2,\text{red}} = 15.131° \text{ and } \theta'_{2,\text{violet}} = \Phi - \theta_{2,\text{violet}} = 15.261°$$

Snell's law at the second interface $\sin\theta_1' = n\sin\theta_2'$ gives the angles for θ_1'.

$$\theta'_{1,\text{red}} = \sin^{-1}\left(n_{\text{red}}\sin\theta'_{2,\text{red}}\right) = 23.212° \text{ and } \theta'_{1,\text{violet}} = \sin^{-1}\left(n_{\text{violet}}\sin\theta'_{2,\text{violet}}\right) = 23.748°$$

The difference of the two θ_1' gives the angular separation of the outgoing rays.

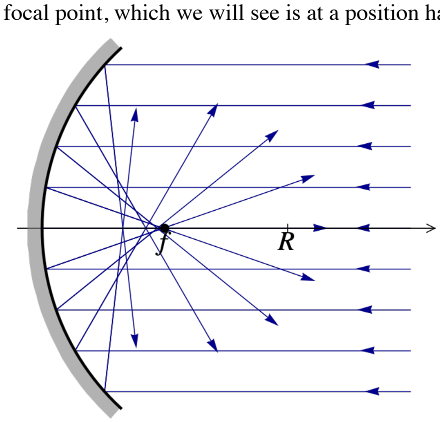
$$\theta'_{1,\text{violet}} - \theta'_{1,\text{red}} = 0.54°$$

K.3 - Images from Spherical and Parabolic Mirrors

Concave Mirrors

Spherical Mirrors and Spherical Aberration

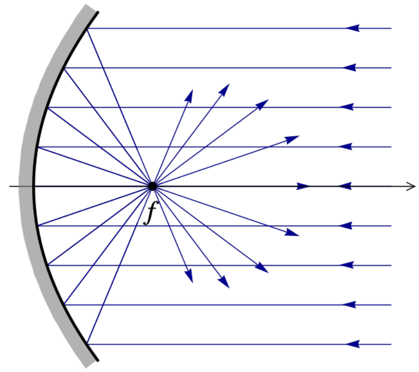
When light from infinity (a plane wave) parallel to the central axis of a concave spherical mirror reflects off the mirror, the rays near the central axis converge toward a point called the focal point, which we will see is at a position half the radius from the mirror.



The rays far from the central axis miss the focal point. This is called spherical aberration.

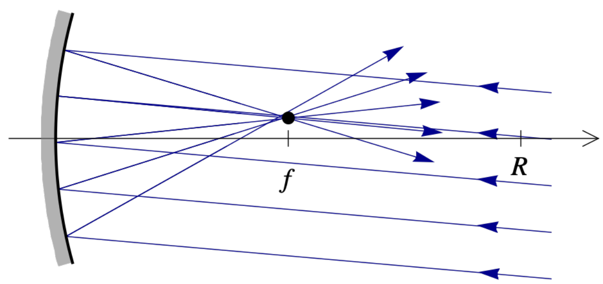
Parabolic Mirrors

The geometrical shape that causes all rays to converge to a single focal point is a paraboloid, which is a parabola under rotation.

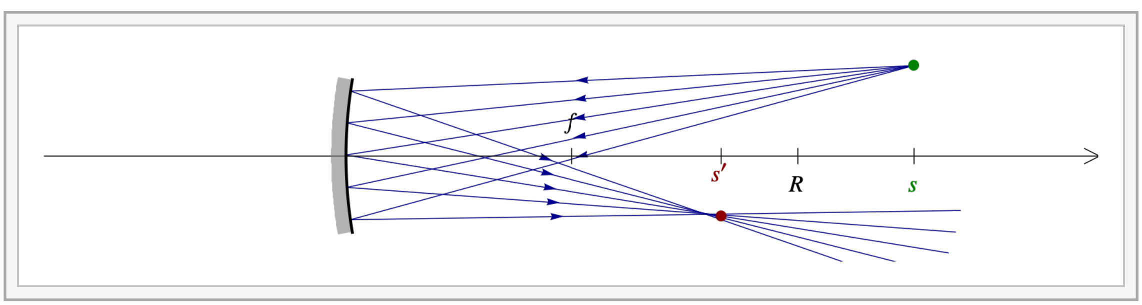


As long as the distance from the central axis is small (compared to the sphere's radius) the sphere approximates a paraboloid well. This is the basic assumption of our analysis of spherical mirrors.

Plane Waves from off the Central Axis

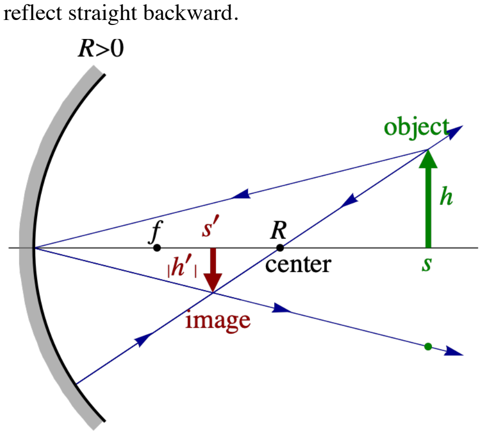


Point Source off the Central Axis

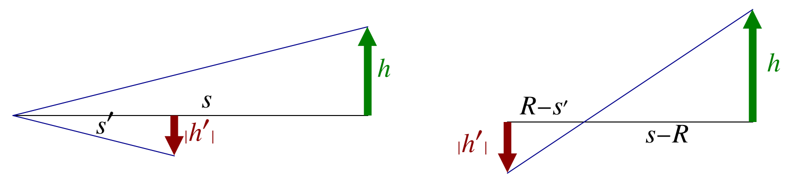


Images from Concave Mirrors

We need to mathematically relate the image position s' and image height h' to the object position s and object height h for the case of a concave spherical mirror. In the preceding diagram it is clear that *every* ray from the tip of the object converges to the tip of the image. It suffices to draw just two rays to find the image position. One ray we will draw is from the tip of the object to the point where the central axis hits the mirror; this will reflect back at an equal angle below the central axis. The other ray we will consider passes through the center of the sphere; this will hit the surface normally and reflect straight backward.



These two rays give two pairs of similar triangles.



These give the expressions

$$\frac{|h'|}{h} = \frac{s'}{s} \text{ and } \frac{|h'|}{h} = \frac{R-s'}{s-R}$$

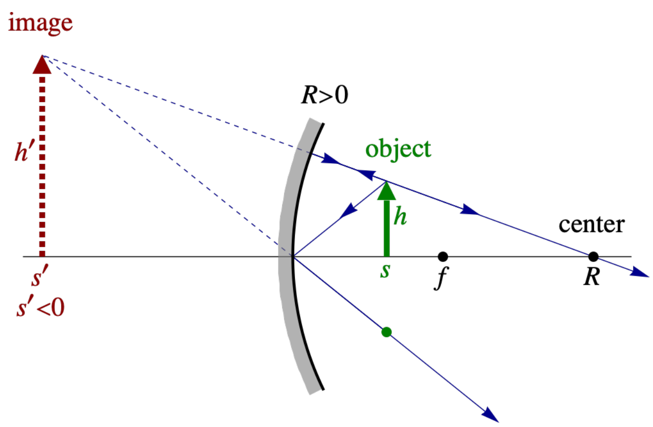
$$\frac{s'}{s} = \frac{R-s'}{s-R} \implies \frac{1}{s} + \frac{1}{s'} = \frac{2}{R}$$

If the object is at infinity ($s \to \infty$) the image is at the focal point ($s' = f$). This gives $f = R/2$. Since the image is inverted we choose, by convention, that $h' < 0$ and thus $|h'| = -h'$. We can rewrite the above expressions as

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \text{ and } m = \frac{h'}{h} = -\frac{s'}{s}.$$

These expressions will apply generally to spherical mirrors, either concave or convex, and to thin lenses, either converging or diverging. m is defined as the magnification; it has the same sign convention as the image height h'.

The preceding image is called a *real image*. A real image occurs when the light rays converge to a point. If the object is inside the focal point, it turns out that the reflected rays do not converge. We trace the same rays as before but to find the image we trace the diverging rays backward to see where the reflected light rays appear to originate; this is the image position. Mathematically, when the object is inside the focal point $0 < s < f$ the image position is negative $s' < 0$.



Example K.6 - Concave Spherical Mirror

A concave spherical mirror with a 60-cm radius is used with a 10-cm high object.

(a) If the object is 40 cm from the mirror, then where is the image and what is the image height and magnification? Also, sketch the rays.

Solution

We are given the radius, which then gives the focal length, the object height and object distance.

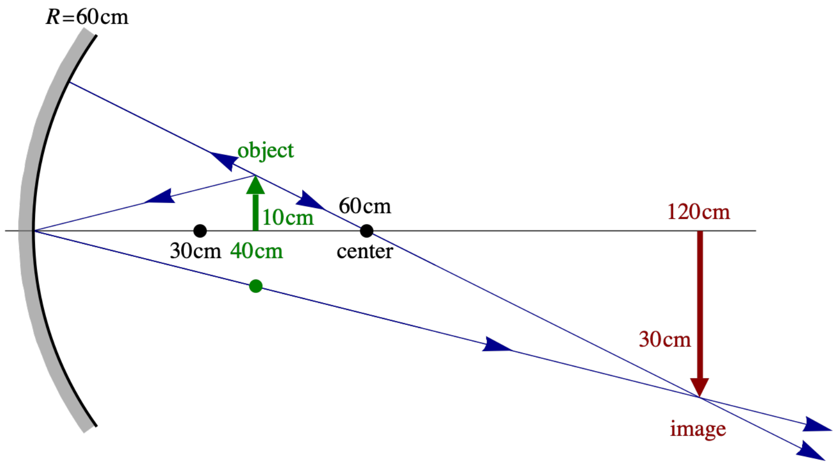
$$R = 60\text{ cm} \implies f = \frac{R}{2} = 30\text{ cm}\,,\quad h = 10\text{ cm and } s = 40\text{ cm}$$

With this we can solve for the image position.

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \implies s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = 120\text{ cm}$$

We can then solve for the image height and magnification.

$$m = \frac{h'}{h} = -\frac{s'}{s} \implies h' = -h\,\frac{s'}{s} = -30\text{ cm} \quad\text{and}\quad m = -\frac{s'}{s} = -3$$



Some comments:

- With as single mirror (or lens as we will see later) a real image is always inverted and inverted images are always real. Inverted images have negative h' and m.
- We can use any length unit (cm, m, ft, in, ...) as long as we are consistent.

(b) If the object is 20 cm from the mirror, then where is the image and what is the image height and magnification? Also, sketch the rays.

Solution

We are given the radius, which then gives the focal length, the object height and object distance.

$$R = 60\text{ cm} \implies f = \frac{R}{2} = 30\text{ cm}\,,\quad h = 10\text{ cm} \;\text{ and }\; s = 20\text{ cm}$$

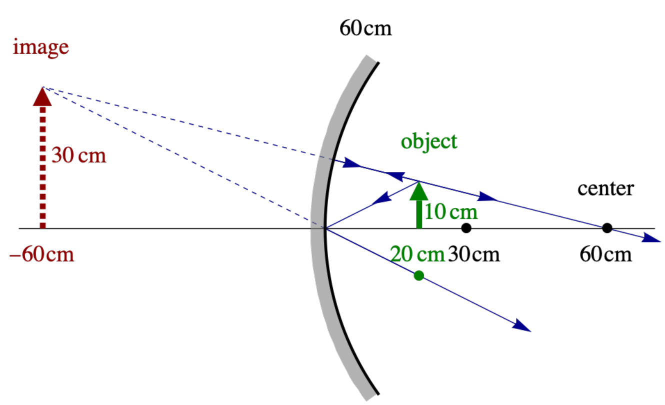
With this we can solve for the image position.

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \implies s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = -60\text{ cm}$$

We can then solve for the image height and magnification.

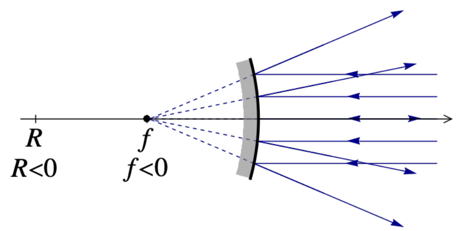
$$m = \frac{h'}{h} = -\frac{s'}{s} \implies h' = -h\,\frac{s'}{s} = +30\text{ cm} \quad\text{and}\quad m = -\frac{s'}{s} = +3$$

This is an upright virtual image.

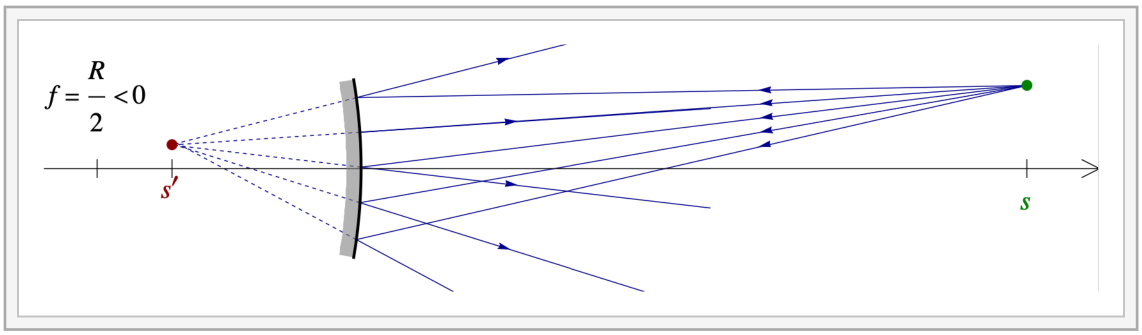


Images from Convex Mirrors

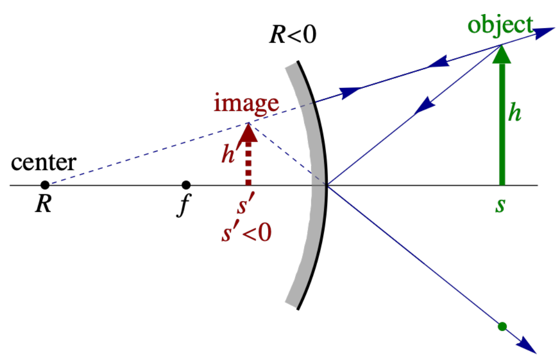
The center of a convex mirror is behind the mirror. We take the radius to be negative. An incident plane wave parallel to the central axis will reflect away from a focal point behind the mirror. When the reflected rays are extrapolated backward they meet at the focal point. We also take the focal length to be negative.



Light from an off-axis point source will reflect away from a point behind the mirror, the image position. This will create a virtual image.



The ray tracing is similar to the concave case except that the center is now behind the mirror.



An analogous analysis with similar triangles yields the same expressions

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \quad \text{and} \quad m = \frac{h'}{h} = -\frac{s'}{s}.$$

It is still true that $f = R/2$ but now both R and f are negative. It follows that since $s > 0$ we will always get $s' < 0$ and thus a virtual image.

Example K.7 - Convex Spherical MIrror

A convex spherical mirror with a 60-cm radius is used with a 10-cm high object. If the object is 20 cm from the mirror, then where is the image and what is the image height and magnification? Also, sketch the rays.

Solution

We are given the magnitude of the radius and we have to add its sign by hand. This then gives the focal length. The object height and object distance are also given.

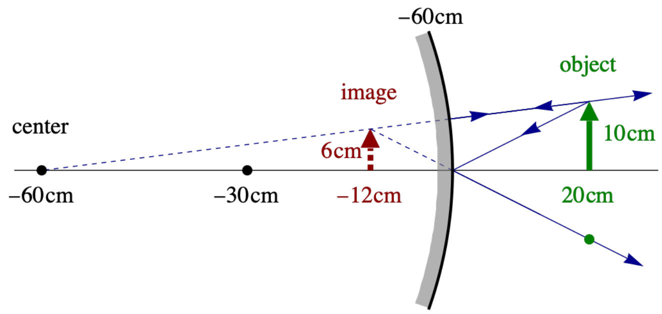
$$R = -60 \text{ cm} \implies f = \frac{R}{2} = -30 \text{ cm}\,, \quad h = 10 \text{ cm} \text{ and } s = 20 \text{ cm}$$

With this we can solve for the image position.

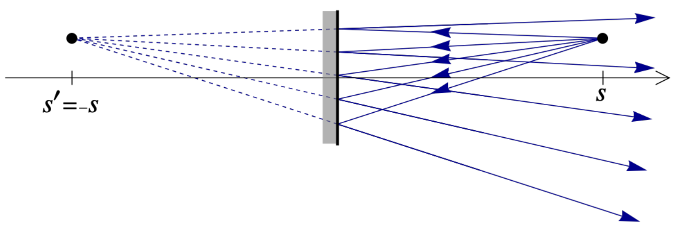
$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \implies s' = \left(\frac{1}{f} - \frac{1}{s}\right)^{-1} = -12\,\text{cm}$$

We can then solve for the image height and magnification.

$$m = \frac{h'}{h} = -\frac{s'}{s} \implies h' = -h\frac{s'}{s} = +6\,\text{cm} \;\text{ and }\; m = -\frac{s'}{s} = +0.6$$



Flat Mirrors

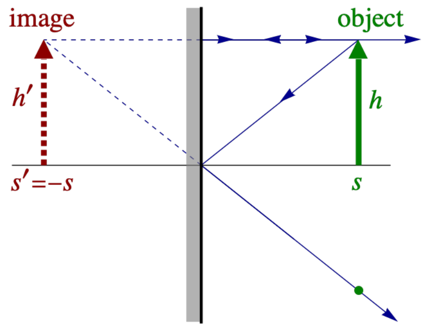


A flat mirror can be viewed as a special case of a spherical mirror with $R \to \infty$. This implies that $f \to \infty$ and $1/f \to 0$

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} = 0 \implies s' = -s$$

$$m = \frac{h'}{h} = -\frac{s'}{s} \implies m = 1 \;\text{ and }\; h' = h$$

This is what we would expect: There is an upright virtual image the same size as the object and equal distance behind the mirror.



K.4 - Images from Refraction

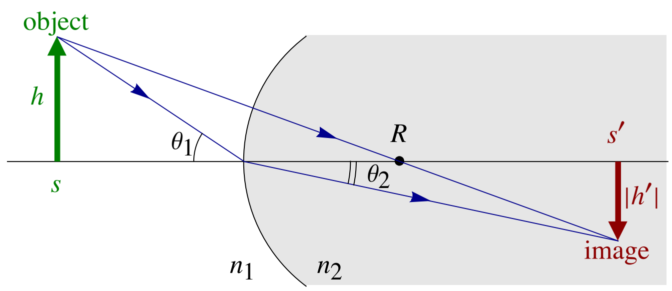
General Sign Conventions in Optics

In the case above where images were formed by mirrors, the sign conventions were relatively simple. s is positive when the object is on the front side of the mirror. Similarly, s', f and R are positive when the image, focal point and center, respectively, are on the mirror's front side.

We will now consider cases involving refraction, where the light emerges (ends up) on the opposite side where it originates. The ray that comes into the optical component (mirror, lens or refracting interface) will be called the incident ray. The ray leaving the optical component will be called the *emergent ray*; this is the reflected ray for mirrors or the refracted ray for a refracting interface or lens.

A general sign convention can now be given. s is positive when the object is on the side of the incident ray, the side where the light originates. s', f and R are positive when the image, focal point and center, respectively, are on the side of the emergent ray, on the side where the light ends up.

Spherical Interface



The underlying assumption is that all angles are small and all rays hit the interface at a small distance from the central axis, relative to the radius. When ϕ is a small angle: $\tan\phi \simeq \phi \simeq \sin\phi$; the $\simeq$ will be replaced with $=$ in this discussion.

$$\sin\,\theta_1 = \tan\,\theta_1 = \frac{h}{s} \text{ and } \sin\theta_2 = \tan\,\theta_2 = \frac{|h'|}{s'} = -\frac{h'}{s'}$$

Snell's law relates the heights to the distances and indices.

$$n_1 \sin\theta_1 = n_2 \sin\theta_2 \implies n_1 \frac{h}{s} = n_2 \frac{|h'|}{s'} \implies \frac{|h'|}{h} = \frac{n_1}{n_2}\frac{s'}{s}$$

We then have a pair of similar triangles, one connecting the object and the center (at R) and the other between the image and center.

$$\frac{|h'|}{h} = \frac{s' - R}{s + R}$$

Combine the two preceding expressions to eliminate $|h'|/h$, cross multiply and then regroup.

$$\frac{n_1}{n_2}\frac{s'}{s} = \frac{s' - R}{s + R}$$
$$\implies n_1\, s\, s' + n_1\, s'\, R = n_2\, s\, s' - n_2\, s\, R$$
$$\implies n_1\, s'\, R + n_2\, s\, R = n_2\, s\, s' - n_1\, s\, s'$$

Dividing by $s\,s'\,R$ gives the expression we seek.

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R}$$

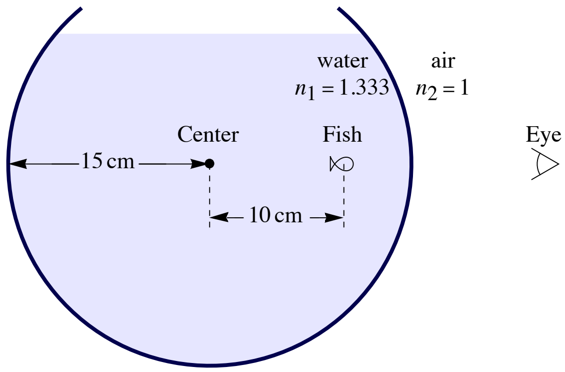
Since the image is inverted we have $|h'| = -h'$. This gives expressions for the image height and magnification.

$$m = \frac{h'}{h} = -\frac{n_1\, s'}{n_2\, s}$$

In the diagram s, s' and R are all positive. The sign convention of R is simple; if the center is on the side of the refracted light, where the index is n_2, then it is positive. If the center is on the side of the incident rays, where the index is n_1, then it is negative. Just as before a positive s' implies a real image and negative means a virtual one.

Example K.8 - A Goldfish in a Bowl

A goldfish is inside a thin glass spherical bowl with a radius of 15 cm and a distance of 10 cm from the center of the bowl, between the center and the eye of the viewer outside the bowl. To the viewer, where is the image of the goldfish? Take the index of water to be 1.333 and since the glass is thin, there is no net refraction due to the glass and it behaves like a direct water to air interface.



Solution

The light passes from the fish through the interface to the eye of the observer. The distance from the fish to the surface is $s = 15\text{ cm} - 10\text{ cm} = 5\text{ cm}$. Since the center of the spherical interface is opposite the side of the emergent ray, the radius is negative.

$$n_1 = 1.333,\ n_2 = 1,\ s = 5\text{ cm and } R = -15\text{ cm}$$

We are looking for s'.

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} \implies \frac{1.333}{5\text{ cm}} + \frac{1}{s'} = \frac{-0.333}{-15\text{ cm}} \implies s' = \left(\frac{0.333}{15\text{ cm}} - \frac{1.333}{5\text{ cm}}\right)^{-1} = -4.09\text{ cm}$$

The negative sign is what we would expect; it is a virtual image, which means that the image of the goldfish is inside the bowl. A real image ($s' > 0$) would have the image of the fish floating midair outside of the bowl.

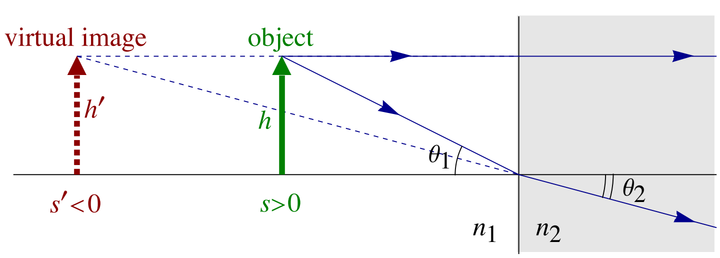
Flat Interface

For a flat interface we let $R \to \infty$ as we did with mirrors.

$$\frac{n_1}{s} + \frac{n_2}{s'} = \frac{n_2 - n_1}{R} = 0 \implies \frac{s'}{s} = -\frac{n_2}{n_1}$$

The magnification, then, becomes one.

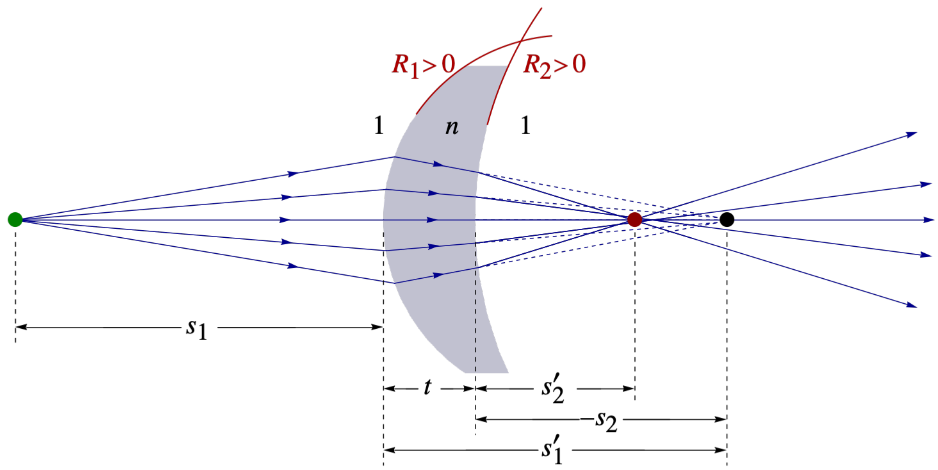
$$m = \frac{h'}{h} = -\frac{n_1}{n_2}\frac{s'}{s} = -\frac{n_1}{n_2}\left(-\frac{n_2}{n_1}\right) \implies m = \frac{h'}{h} = 1$$



Notice from the diagram above that it is clear that $m = 1$. Also, the diagram shows this can only be a virtual image.

Thin Lens Approximation

A lens is formed from two refracting surfaces. The light passes from air ($n = 1$) to a medium with index n and then back to air. Take the two refracting surfaces to have radii R_1 and R_2. Using the standard sign convention, take both radii to be positive; this means that their centers are on the side of the light leaving the lens.



Begin with an object to the left of our lens, shown by the green dot; its distance from the first surface is s_1. At the first interface we have

$$\frac{1}{s_1} + \frac{n}{s_1'} = \frac{n-1}{R_1}.$$

This gives an image beyond the first surface by s_1' shown as the black dot. Note that this image is also beyond the second surface. When there is a combination of optical elements, in this case refracting surfaces, the image from the first becomes the object for the second. Since the image from the first surface has yet to form when it hits the second we get what is called a virtual object; this is the odd circumstance when s becomes negative. s_2 is measured from the second interface; its negative value is:

$$t - s_2 = s_1' \implies s_2 = t - s_1'.$$

At the second interface we have

$$\frac{n}{s_2} + \frac{1}{s_2'} = \frac{1-n}{R_2},$$

where the image, shown with the red dot, is a distance of s_2' beyond the second surface.

To make this a thin lens we let $t \to 0$, giving $s_2 = -s_1'$. Also, take the image and object distances for the lens to be the incoming object distance and the outgoing image distance.

$$s_2 = -s_1',\ s = s_1 \text{ and } s' = s_2'$$

Combining all these expressions above gives an expression relating the object and image positions to the index of the lens' material and the radii of curvature of its surfaces.

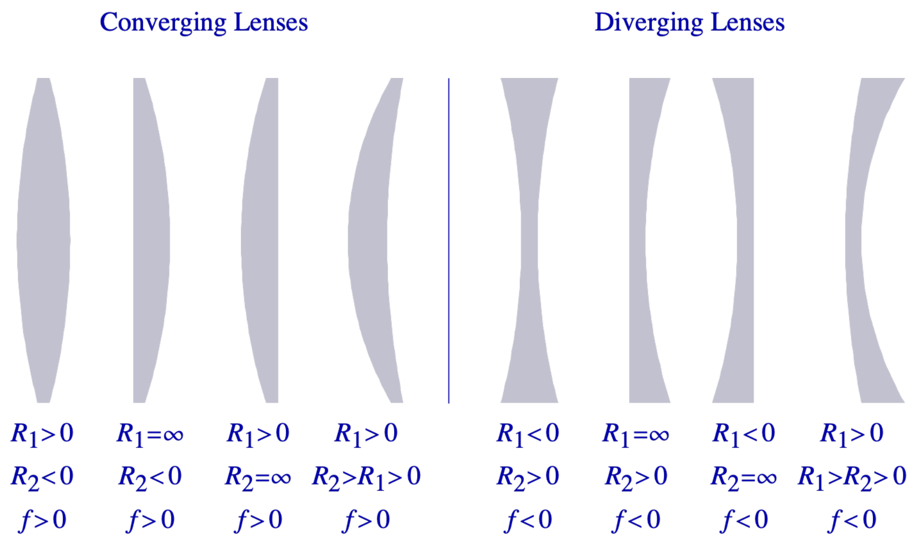
$$\frac{1}{s} + \frac{1}{s'} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right)$$

The right-hand side can be identified with $1/f$, where f is defined as the focal length of the lens. This gives what is called the *lensmaker formula*:

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

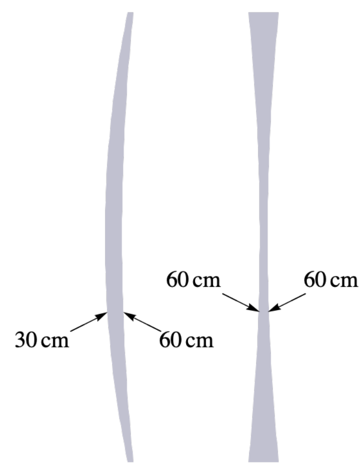
With this identification the same expressions that applied to both the concave and convex spherical mirrors also apply to the case of a thin lens

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \text{ and } m = \frac{h'}{h} = -\frac{s'}{s}.$$



Lenses with $f > 0$ are called converging lenses and those with $f < 0$ are called diverging. Note that converging lenses are always thicker at the center and diverging are thinner at the center.

Example K.9 - The Lensmaker Formula and Eyeglasses



(a) (b)

The figure above shows two lenses for eyeglasses that use a plastic lens with $n = 1.6$. The lens on the left is converging and for the farsighted and the lens on the right is diverging and for the nearsighted.

(a) What is the focal length of each lens?

Solution

This will use the lensmaker formula.

$$\frac{1}{f} = (n-1)\left(\frac{1}{R_1} - \frac{1}{R_2}\right).$$

The signs of the radii are not given. The sign convention is that the radius is positive when the center is on the side where the light emerges from the lens, take this to be the right side. For part (a) both are positive

$$\text{(a)}\ n = 1.6,\ R_1 = 30\ \text{cm and } R_2 = 60\ \text{cm} \implies f = 100\ \text{cm}$$

and for part (b) the surface ont eh left the radius is negative and the surface on the right is positive.

$$\text{(b)}\ n = 1.6,\ R_1 = -60\ \text{cm and } R_2 = 60\ \text{cm} \implies f = -50\ \text{cm}$$

(b) In eyeglass prescriptions, the "lens power" is given in diopters. This is related to the focal length:

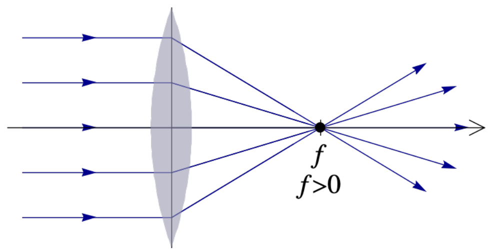
$$\text{Lens Power (in Diopters)} = \frac{100}{f\,(\text{in cm})}$$

What is the “lens power” of both lenses in diopters?

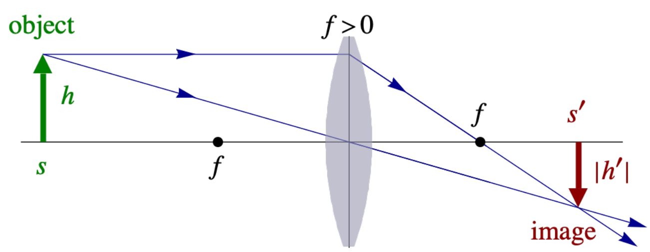
Solution

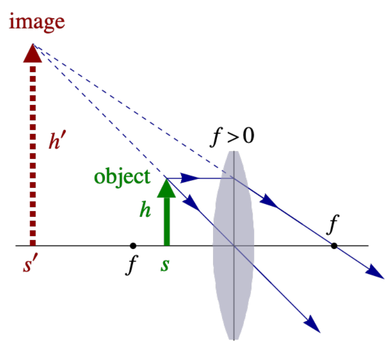
$$\text{(a)}\quad f = 100\text{ cm} \implies +1\text{ diopters}$$
$$\text{(b)}\quad f = -50\text{ cm} \implies -2\text{ diopters}$$

Converging Lenses

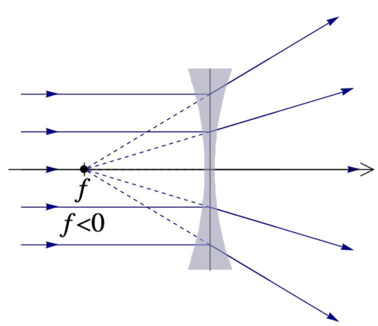


A thin lens with a positive focal length is called a converging lens. This is analogous to a concave mirror mathematically. If the object distance is larger than the focal length $s > f$ then there is a real image and when the object distance is less than the focal length $s < f$ there is a virtual image. To trace the rays: Draw one ray that passes straight through where the central axis hits the lens. Draw the other ray parallel to the central axis; this will bend through the focal point. The two rays (as all rays do) will converge to the image. Below are examples of ray tracing in both cases.

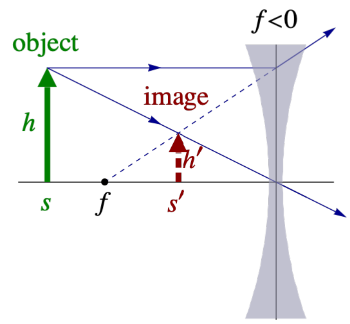




Diverging Lenses



A diverging lens has a negative focal length. The ray tracing is similar but the ray that begins parallel to the central axis will diverge away from the focal point, which is on the same side as the object. Here is the ray tracing for a diverging lens.



Example K.10 - A Slide Projector

A slide projector uses a bright light to shine through a small slide. The slide is the object and it is projected onto a screen using a converging lens. For a focused image, the image is on the screen. Note that this is the same optics as a classroom projector, where there is a small backlit LCD screen that is focused, via a converging lens, to a screen.

A slide projector uses a converging lens with a 12-cm focal length to project a 35-mm wide slide to fill a 2.1-m wide screen. Relative to the lens, where must the slide and screen be placed?

Solution

We are given the focal length $f = 12\,\text{cm}$ and the object and image heights. (They are described as widths, but that is unimportant.) $h = 35\,\text{mm} = 3.5\,\text{cm}$ and $|h'| = 2.1\,\text{m} = 210\,\text{cm}$. The use of the absolute value for h' is a bit subtle. We will see that the image is inverted but a positive h' will give us a clear inconsistency.

$$f = 12\,\text{cm}\,, \quad h = 3.5\,\text{cm} \;\text{ and }\; h' = \pm 210\,\text{cm}.$$

A projected image must be a real image.

$$\frac{h'}{h} = -\frac{s'}{s} \implies s' = -\frac{h'}{h}\,s$$

For a single lens or mirror, the object position s must be positive and a real image must have a positive s'. If both h and h' are positive then d_i is negative and we have an inconsistency.

$$h' = -210\,\text{cm} \implies s' = -\frac{h'}{h}\,s = 60\,s$$

We now have two equations for the two unknowns s and s'

$$\frac{1}{s} + \frac{1}{s'} = \frac{1}{f} \implies \frac{1}{s} + \frac{1}{60\,s} = \frac{1}{12\,\text{cm}}$$

Cross-multiplying gives

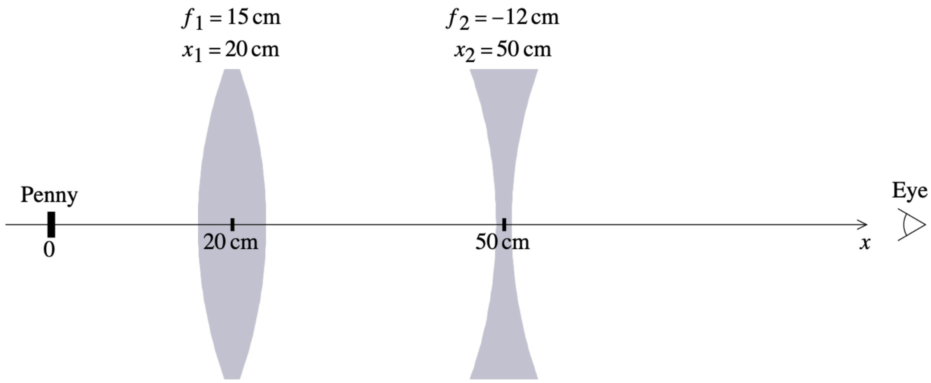
$$12\,\text{cm}\left(1 + \frac{1}{60}\right) = s$$

We can then solve for s and s'.

$$s = 12.2\,\text{cm} \;\text{ and }\; s' = 60\,s = 732\,\text{cm}$$

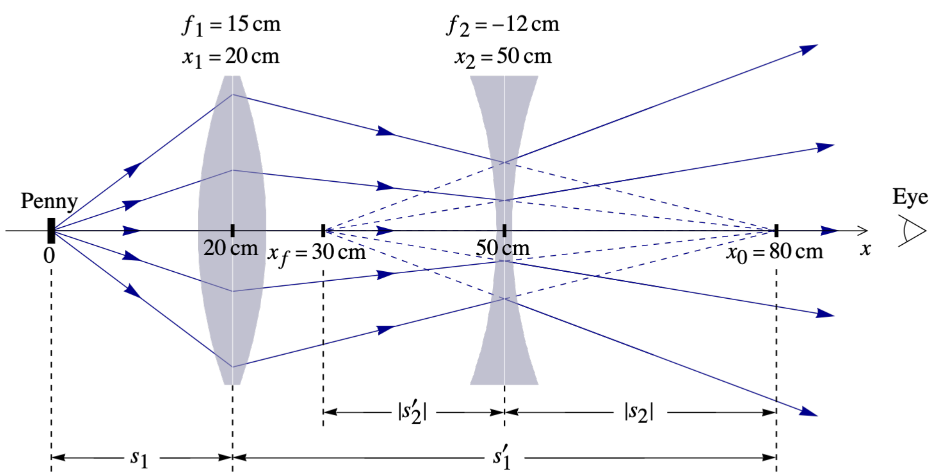
Example K.11 - Two Lenses and a Virtual Object

A penny at $x = 0$ is viewed through two lenses, a converging lens with $f_1 = 15\,\text{cm}$ at $x_1 = 20\,\text{cm}$ and a diverging lens with $f_2 = -12\,\text{cm}$ at $x_2 = 50\,\text{cm}$. Where is the final image of the penny after passing through both lenses? Give the x-coordinate of its position.



Solution

When you have two lenses (or more generally two optical components: mirrors, lenses or refracting surfaces) the image from the first lens is the object for the second. Suppose in the arrangement above the image distance from the first lens was $s_1' = 25\,\text{cm}$, then that would be at $x = 45\,\text{cm}$; this is $5\,\text{cm}$ before the second lens, so $s_2 = 5\,\text{cm}$. However, we will see that actually with this problem $s_1' = 60\,\text{cm}$ at $x_0 = 80\,\text{cm}$, which is beyond the second lens. The light approaching the second lens is not moving away from an object before it but instead moving toward a virtual object beyond it. This virtual object will have $s_2 = -30\,\text{cm}$.



The light from the penny passes through the converging lens, the diverging lens and to the eye of the viewer. First we consider the converging lens. The penny (the object) is 20 cm from the lens.

$$s_1 = x_1 = 20\,\text{cm} \text{ and } f_1 = 15\,\text{cm} \implies \frac{1}{s_1} + \frac{1}{s_1'} = \frac{1}{f_1} \implies s_1' = \left(\frac{1}{f_1} - \frac{1}{s_1}\right)^{-1} = 60\,\text{cm}$$

This image is 60 cm beyond (to the right of) the lens at 20 cm, so it is at the position x_0.

$$x_0 = x_1 + s_1' = 20\,\text{cm} + 60\,\text{cm} = 80\,\text{cm}$$

The image from the first lens becomes the object for the second. This means that x_0 the *position* of the image from the first lens is the *position* of the object for the second. Because this is beyond the diverging lens at $x_2 = 50\,\text{cm}$ by $30\,\text{cm}$, it is a virtual object with $s_2 = -30\,\text{cm}$

$$s_2 = x_1 - x_0 = 50\,\text{cm} - 80\,\text{cm} = -30\,\text{cm}$$

With this we can find the image distance for the second lens s_2'.

$$f_2 = -12\,\text{cm} \implies \frac{1}{s_2} + \frac{1}{s_2'} = \frac{1}{f_2} \implies s_2' = \left(\frac{1}{f_2} - \frac{1}{s_2}\right)^{-1} = -20\,\text{cm}$$

The coordinate of the final image position x_f is then

$$x_f = x_2 + s_2' = 50\,\text{cm} + (-20\,\text{cm}) = 30\,\text{cm}$$

and this is our answer.